ADVANCED ALGEBRA

VOLUME I

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BY

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PREFACE

THIS book was originally announced as forming Part IV of the author's *New Algebra for Schools*, which contains in Parts I-III all the algebra required for School Certificate, including "additional mathematics." As, however, this part is essentially one for advanced students, it has been decided to issue it as the first volume of an "Advanced Algebra," which will complete in a second volume the school course for mathematical specialists.

The present volume deals comprehensively with Higher Certificate work up to, but excluding, the special requirements for "distinction papers" and the higher algebra demanded for entrance scholarships at the Universities. The second volume is being written in collaboration with Mr. A. Robson and forms a companion text-book to the *Advanced Trigonometry* by the same authors.

In accordance with modern practice, calculus methods have been employed whenever the treatment of the subject is simplified thereby, for example, in the summation of certain types of finite series and in the study of the properties of rational functions. In addition, experience has proved that the most intelligible approach to the logarithmic and exponential functions is along calculus lines, and valid proofs of the binomial theorem and the expansions of $\log(1+x)$ and e^x (if valid proofs are required) can be obtained far more simply by calculus methods than by the old-fashioned algebraic treatment. For the benefit of mathematical specialists, such proofs are given in this book, but no doubt many students will be content to confine themselves to applications of the general theory.

The treatment of limits and convergence in Chapter IV is limited to what is suitable for a first reading; it is undesirable, and indeed impracticable, to develop at the stage at which this should be taken the formal ϵ, δ technique which the specialist will tackle later; but the object of the chapter is to show clearly what is the nature of a limit and what general considerations should be

in the mind of the student when considering whether a limit does or does not exist.

The chapter on determinants has been put last, but may be taken at any stage of the course and should be read as soon as the student's work in analytical geometry requires the use of this notation.

The author is deeply indebted to Mr. A. Robson and to Mr. J. C. Manisty for numerous criticisms and suggestions.

C. V. D.

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CONTENTS

ADVANCED ALGEBRA

CHAPTER I

PERMUTATIONS AND COMBINATIONS

The r, s Principle

Example 1. There are 5 ways from A to B and 3 from B to C; how many are there from A to C via B?

Any one of the 5 ways from A to B can be combined with any one of the 3 ways from B to C.

$\therefore$ there are 5×3 different routes from A to C via B.

The principle used in this example may be stated as follows:

If one operation can be performed in r different ways, and if a second operation can then be performed in s different ways, the two operations can be performed in succession in $r \times s$ different ways.

Example 2. How many different arrangements can be made of the 4 letters, a, b, c, d?

Any one of the 4 letters can be put first. When the first place is filled, 3 letters remain, any one of which can be put second.

$\therefore$ the first two places can be filled in 4×3 ways.

2 letters remain, either of which can be put third, and 1 letter then remains for the fourth place.

$\therefore$ the number of different arrangements is $4 \times 3 \times 2 \times 1$.

This example illustrates the principle that if one operation can be performed in r different ways, and if a second operation can then be performed in s different ways, and if a third operation can then be performed in t different ways, the three operations can be performed in succession in $r \times s \times t$ different ways.

The principle can evidently be extended to any number of operations performed in succession.

Arrangements with Repetitions

Example 3. Residents at a boarding-house can choose *either* fish *or* eggs *or* bacon *or* sausages for breakfast. In how many ways can a man arrange his breakfasts for a week?

He has 4 choices on Sunday, 4 on Monday, 4 on Tuesday, and so on.

$\therefore$ the number of different arrangements for a week is

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7.$$

The number of arrangements of n unlike things, r at a time, when each may be repeated any number of times, is $\mathbf{n^r}$.

Imagine r compartments in a row, each of which can hold one of the unlike things, but no more.

The first compartment can be filled in n ways. And, as repetitions are allowed, the second can also be filled in n ways, and so on.

$\therefore$ the number of different arrangements is

$$n \times n \times n \times \dots r \text{ factors} = \mathbf{n^r}.$$

Factorials

The index notation makes it possible to write the result of Ex. 3 in a concise form. It is often desirable to have a short way of expressing the product of a number of consecutive integers. The product of the first n positive integers, $1 \times 2 \times 3 \times 4 \times \dots \times n$ is denoted by $n!$ or $\underline{|n}$ and is called "**factorial n.**"

Thus the answer to Example 2, p. 1, could be given as 4!. The product of any number of consecutive integers can be represented as the quotient of two factorials.

Thus $$9 \times 8 \times 7 \times 6 = \frac{9!}{5!}.$$

More generally, the product of r consecutive integers, of which n is the greatest, can be written

$$n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}.$$

Arrangements of Unlike Things in Line

The number of ways of arranging n unlike things all in a row is n!

Any one of the n things can be put first. When the first place has been filled, $(n-1)$ things remain, any one of which can be put second.

$\therefore$ the first two places can be filled in $n(n-1)$ ways.

When the first two places have been filled, $(n-2)$ things remain, any one of which can be put third.

$\therefore$ the first three places can be filled in $n(n-1)(n-2)$ ways; and so on.

$\therefore$ the total number of ways of arranging the n unlike things in a row is

$$n(n-1)(n-2)(n-3)\ldots 3.2.1 = n!.$$

Circular Arrangements of Unlike Things

The number of ways of arranging n unlike things round a circle, regarding clockwise and counterclockwise arrangements as different, is $(n-1)!$.

Since the order round the circle is all that matters, we can choose one special thing and keep it always in the same place. The remaining $(n-1)$ things can then be arranged in $(n-1)!$ ways.

If no distinction is made between clockwise and counterclockwise arrangements, the total number is half this amount, namely

$$\frac{(n-1)!}{2}.$$

Thus 6 men can be arranged at a round table in $5!$ ways; but if 6 beads of unlike colours are threaded on a ring, there are only $\frac{5!}{2}$ different designs.

Example 4. In how many ways can 10 different books be arranged on a shelf so that two particular books are next to one another?

Imagine the two particular books fastened together; this can be done in 2 ways, as either may come first.

There are now 9 unlike things to be arranged in a row, and this can be done in $9!$ ways.

$\therefore$ the number of arrangements is $9! \times 2$.

EXERCISE I. a

1. In how many ways can a boy and girl be chosen from 6 boys and 9 girls ?

2. In how many ways can 5 boys take their places on a bench ?

3. In how many ways can 3 different prizes be awarded to 10 boys, if any boy may win them all ?

4. In how many ways can a first and second prize be awarded in a class of 10 boys ?

5. In how many ways can a first, second and third prize be awarded in a class of 10 boys ?

6. In how many ways can I choose 1 novel, 1 magazine and 1 newspaper from 12 novels, 5 magazines and 8 newspapers ?

7. How many code words of 4 letters can be formed from the 26 letters of the alphabet ?

8. There are 5 gramophone records with a dance tune on each side of each record. In how many orders can the tunes be played, none being repeated ? How many arrangements are possible if there is only time to play 4 tunes ?

9. Find the values of

$$\text{(i) } 6!; \quad \text{(ii) } \frac{7!}{4!}; \quad \text{(iii) } \frac{8!}{5!\,3!}.$$

10. Express in factorials :

(i) $10 \times 9 \times 8$; (ii) $10 \times 11 \times 12 \times 13$;
(iii) $n(n-1)(n-2)$; (iv) $n(n+1)(n+2)(n+3)$;
(v) $n(n^2-1)(n^2-4)(n^2-9)$;
(vi) $n(n+1)(n+2)(n+3) \dots (n+r)$.

11. In how many orders can the letters A, B, C, D, E be marked round a circle drawn on the blackboard ?

12. Every day can be described as either fine or wet or indifferent. Within how many years must there be a repetition of the description of a week's weather ? (A week begins on Sunday.)

13. In how many orders can the letters of the word *treason* be arranged ? How many arrangements begin with t ? How many begin with t and end with n ?

14. In how many ways can 8 boys be arranged in a row ? In how many of these ways do 2 particular boys occupy end places ?

15. In how many ways can 6 people be arranged at a round table so that 2 particular people sit together ?

16. How many even numbers of 3 digits can be formed with the figures 4, 5, 6, (i) if no figure is repeated, (ii) if repetitions are allowed ?

17. Simplify (i) $n! \div (n-1)!$; (ii) $n! - (n-1)!$

18. In how many orders can the letters of the word *Sunday* be arranged? How many of these arrangements do not begin with *s*? How many begin with *s* and do not end with *y*?

19. There are n stations on a local railway line. How many different kinds of single third-class tickets must be printed if it is possible to book from any one station to any other?

20. Prove that $(2n)! \div n! = 2^n . \{1 . 3 . 5 . 7 \ldots (2n-1)\}$.

21. Prove that $2 . 6 . 10 . 14 \ldots (4n-6)(4n-2)$

equals $(n+1)(n+2)(n+3) \ldots (2n-1)2n$.

22. If $(mn+1)$ pairs of numbers are written down, each pair consisting of one chosen from the m letters $a_1, a_2, \ldots a_m$, and one from the n letters $b_1, b_2, \ldots, b_n$, prove that at least two of the pairs are identical.

Notation for Number of Permutations

Example 5. How many arrangements can be made with the 26 letters of the alphabet, if each contains 3 different letters?

Imagine three compartments in a row.

First letter	Second letter	Third letter

Any one of the 26 letters can be put in the first compartment. When this has been done, 25 letters remain, any one of which can be put in the second compartment.

$\therefore$ the first two compartments can be filled in 26×25 ways.

24 letters now remain, any one of which can be put in the third compartment.

$\therefore$ the total number of arrangements is $26 \times 25 \times 24$.

This is called "the number of permutations of 26 things, taken 3 at a time," and is written ${}_{26}P_3$.

More generally, the number of arrangements that can be made with n *unlike* things, if each arrangement contains r of them, is called **the number of permutations of n things, taken r at a time,** and is denoted by ${}_nP_r$.

In particular, ${}_nP_n$ denotes the number of ways of arranging n unlike things, all being taken; this is the number of ways of arranging n unlike things, all in a row, which was shown on p. 3 to be $n!$.

The number of arrangements, or permutations, of n unlike things, taken r at a time, is

$$_n\mathrm{P}_r = n(n-1)(n-2)\dots(n-r+1).$$

The number of arrangements equals the number of ways of putting one thing into each one of a row of r compartments, when n unlike things are available.

The first compartment can be filled in n ways. When this has been done, the second can be filled in $(n-1)$ ways.

$\therefore$ the first two can be filled in $n(n-1)$ ways.

The third can then be filled in $(n-2)$ ways; therefore the first three can be filled in $n(n-1)(n-2)$ ways, and so on.

The rth or last compartment can finally be filled in $n-(r-1)$ ways, that is $(n-r+1)$ ways.

$$\therefore\ _n\mathrm{P}_r = n(n-1)(n-2)\dots(n-r+1).$$

$_n\mathrm{P}_r$ has been defined only for $r = 1, 2, 3, \dots, n$, and except for $r=n$, the expression just obtained for it can be written

$$_n\mathrm{P}_r = \frac{n(n-1)(n-2)\dots(n-r+1)\times(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}.$$

Since $_n\mathrm{P}_n = n!$, this result holds also for $r=n$ if $0!$ is given the value 1; the symbol $0!$ has not been previously defined in this book, and so we now define it to be 1.

Conditional Arrangements

In a few of the examples of Exercise I. a, the required arrangements were subject to very simple restrictions. The following example illustrates the procedure when the conditions are a little more elaborate.

Example 6. How many numbers greater than 7000 can be formed with the digits 3, 5, 7, 8, 9, no digit being repeated?

If the number contains 5 digits, it can be formed in $_5\mathrm{P}_5 = 5! = 120$ ways.

If the number contains 4 digits, the left-hand digit can be 7 or 8 or 9, but not 3 or 5; therefore the left-hand digit can be chosen in 3 ways. Whichever left-hand digit is chosen, the arrangement can be completed in $_4\mathrm{P}_3$ ways.

$\therefore$ the number of arrangements with 4 digits is

$$_4P_3 \times 3 = 4 \,.\, 3 \,.\, 2 \,.\, 3 = 72.$$

$\therefore$ the total number of arrangements is $120 + 72 = 192$.

Like and Unlike Things

The number of ways of arranging n things, all in a row, when there are p alike of one kind, q alike of another kind, r alike of another kind, and so on, is

$$\frac{n!}{p!\, q!\, r! \ldots}.$$

Consider the n unlike letters,

$$a_1, a_2, a_3, \ldots, a_p, b_1, b_2, \ldots, b_q, c_1, c_2, \ldots, c_r, \ldots.$$

The number of ways of arranging n unlike things, all in a row, is $n!$.

Now if any one arrangement is written down, the letters, $a_1, a_2, \ldots, a_p$, can be arranged among themselves, *without altering the positions of any other letters*, in $p!$ ways. But if the suffixes of all the letters a are removed, these $p!$ different arrangements become identical.

$\therefore$ the number of arrangements with p like letters a and all other letters unlike, n letters in all, is $\frac{n!}{p!}$.

Similarly, in any one of these $\frac{n!}{p!}$ arrangements, the letters $b_1, b_2, \ldots, b_q$, can be arranged among themselves, without altering the position of any other letters in $q!$ ways, and if the suffixes of all the letters b are removed, these $q!$ different arrangements become identical.

$\therefore$ the number of arrangements with p like letters a and q like letters b, and all other letters unlike, is $\frac{n!}{p!\, q!}$.

This argument can be repeated as often as necessary.

$\therefore$ the final number of arrangements is $\frac{n!}{p!\, q!\, r! \ldots}$.

Example 7. Find the number of ways of arranging the letters *aaaaabbbccdef* in a row, if the letters b are separated from one another.

The 10 letters *aaaaaccdef* can be arranged in $\frac{10!}{5!\, 2!}$ ways.

In any one of these arrangements, there are 11 places where the letter b_1 can be inserted. When this has been done, there are 10 places where b_2 can be inserted so as not to be next to b_1; and then 9 places where b_3 can be inserted so as not to be next to b_1 or b_2.

$\therefore$ the total number of arrangements of $aaaaaccdefb_1b_2b_3$ when b_1, b_2, b_3 are separated from one another is

$$\frac{10!}{5!\,2!}\times 11\times 10\times 9.$$

$\therefore$ when the suffixes are removed, the number is

$$\frac{10!}{5!\,2!}\cdot\frac{11\times 10\times 9}{3!}=\frac{10!\,11!}{2!\,3!\,5!\,8!}.$$

EXERCISE I. b

1. How many arrangements can be made with the letters, a, b, c, d, e, f, if each contains (i) 2 unlike letters, (ii) 5 unlike letters ?

2. A shelf holds 6 books. In how many ways can it be filled if 10 unlike books are available ?

3. How many numbers of 3 different digits can be formed with the figures 1, 2, 3, 4, 5, 6, 7 ?

4. How many numbers greater than 500 can be formed with the digits 4, 5, 6, 7, repetitions not being allowed.

5. In how many ways can 7 people be arranged at a round table so that the oldest and youngest sit together ?

6. In how many ways can 8 beads of different colours be arranged on a ring ? In how many of these arrangements are the red and yellow beads separated ?

7. Find the values of

$$\text{(i) } {}_5P_2\,;\quad \text{(ii) } {}_6P_3\,;\quad \text{(iii) } {}_5P_5.$$

8. Write down expressions for

$$\text{(i) } {}_nP_2\,;\quad \text{(ii) } {}_nP_5\,;\quad \text{(iii) } {}_{2n}P_n.$$

9. How many even numbers of 4 digits can be formed from the figures 2, 3, 4, 6, if repetitions are allowed ?

10. How many odd numbers above 4000 can be formed from the figures 1, 2, 3, 4, 6, if repetitions are not allowed ?

11. In how many ways can 5 dots and 3 dashes be arranged in a row ?

12. In how many ways can the letters in *rearrange* be arranged ? In how many of these do the letters a come together ?

13. In how many ways can 4 red counters, 4 white counters and 1 black counter be arranged in a row ?

14. In how many ways can 4 boys and 3 girls be arranged in a line so that boys and girls are placed alternately ?

15. In how many ways can the crew of an eight-oared boat be arranged (i) if 4 of the crew can row only on the stroke side, (ii) if 3 of the crew can row only on the stroke side and if 2 can row only on the bow side ?

16. In how many orders can 8 stories be arranged in a book if neither the longest nor the shortest comes first ? In how many of these ways does the longest come last ?

17. How many arrangements can be made of the letters in *photograph* ? In how many of these are there exactly 5 letters between the two letters *h* ?

18. Four travellers arrive at a town where there are 5 hotels. In how many ways can they be lodged (i) if no two go to the same hotel, (ii) if a given pair go to the same hotel, and the others to any of the other hotels, (iii) if there are no restrictions ?

19. In how many ways can 5 men and 2 ladies be arranged at a round table if the two ladies (i) sit together, (ii) are separated ?

20. In how many ways can 5 different Latin books, 4 different Greek books, 3 different French books be arranged on a shelf so that the books in each language come together ?

21. In how many ways can the letters in *deposit* be arranged if the vowels come in the even places ?

22. How many arrangements of the letters in *tomato* are such that the *t*'s are separated ?

23. Find the number of ways in which 10 candidates can be ranked in order of merit if (i) A is next to B, (ii) A is above B ? None are bracketed equal.

24. In how many ways can the 8 seats in a railway carriage be assigned to 8 men, 2 of whom must face the engine and 1 must have his back to the engine ?

25. In how many ways can 6 ladies and 6 gentlemen be arranged at a round table, if two particular ladies must not sit next to one particular man, all the men being separated ?

26. There are 5 *a*'s, 4 *b*'s, 3 *c*'s, 3 *d*'s, and 6 other different letters. In how many ways can they be arranged so that no two *a*'s come together ?

27. In how many different orders can 10 examination papers be set so that no two of the three mathematical papers are consecutive ?

Selections

In calculating the number of selections which can be made from a given set of unlike things, no regard is paid to the order in which the things occur in the chosen group. A change of order of the things in the group gives a new *arrangement*, but does not affect the *selection*.

There are two methods of tackling the general problem: the first treats it *ab initio*; the second is easier but assumes the formula obtained for ${}_nP_r$.

First Method

Example 8. How many selections of 2 letters can be made from the 6 letters, a, b, c, d, e, f?

If we write down all the different selections, each letter (such as a) comes 5 times in the list, namely once with each of the other letters (b, c, d, e, f); hence the total number of letters in the list is 6×5, and therefore the number of selections is $\frac{6 \times 5}{2}$.

This is called the number of **combinations** of 6 things taken 2 at a time and is denoted by ${}_6C_2$ or $\binom{6}{2}$.

Thus $${}_6C_2 = \frac{6 \times 5}{2}.$$

More generally, the number of ways of selecting r things from n unlike things is called the **number of combinations of n things, taken r at a time**, and is denoted by ${}_nC_r$ or $\binom{n}{r}$.

Example 9. How many selections of 3 letters can be made from the 7 letters, a, b, c, d, e, f, g?

If all the ${}_7C_3$ selections are written down, there are $3 \times {}_7C_3$ letters in the list, since each selection contains 3 letters. Each letter (such as a) comes ${}_6C_2$ times in the list since it occurs with each selection of 2 letters from the other 6 letters (b, c, d, e, f, g). Hence the total number of letters in the list is also $7 \times {}_6C_2$.

$$\therefore\ 3 \times {}_7C_3 = 7 \times {}_6C_2; \quad \therefore\ {}_7C_3 = \tfrac{7}{3} \times {}_6C_2.$$

But ${}_6C_2 = \frac{6 \times 5}{2}$, see Example 8,

$$\therefore\ {}_7C_3 = \frac{7}{3} \times \frac{6 \times 5}{2} = \frac{7.6.5}{1.2.3}.$$

The number of selections, or combinations, of n unlike things, taken r at a time, is

$$_n\mathrm{C}_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r} = \frac{n!}{r!(n-r)!}.$$

Denote the n unlike things by the letters, $a_1, a_2, a_3, \dots, a_n$.

If all the $_n\mathrm{C}_r$ selections are written down, there are $r \times {_n\mathrm{C}_r}$ letters in the list, since each selection contains r letters. Each letter (such as a_1) comes $_{n-1}\mathrm{C}_{r-1}$ times in the list since it occurs with each selection of $(r-1)$ letters from the other $(n-1)$ letters, $(a_2, a_3, \dots, a_n)$. Hence the total number of letters in the list is also $n \times {_{n-1}\mathrm{C}_{r-1}}$.

$$\therefore\ r \times {_n\mathrm{C}_r} = n \times {_{n-1}\mathrm{C}_{r-1}}\,; \quad \therefore\ {_n\mathrm{C}_r} = \frac{n}{r} \times {_{n-1}\mathrm{C}_{r-1}}.$$

This relation holds for all values of n and r so long as $n \geqslant r > 1$.

$$\therefore\ {_{n-1}\mathrm{C}_{r-1}} = \frac{n-1}{r-1} \times {_{n-2}\mathrm{C}_{r-2}}\,; \quad {_{n-2}\mathrm{C}_{r-2}} = \frac{n-2}{r-2} \times {_{n-3}\mathrm{C}_{r-3}}\,;$$

and so on, down to $_{n-r+2}\mathrm{C}_2 = \dfrac{n-r+2}{2} \times {_{n-r+1}\mathrm{C}_1}$.

$$\therefore\ {_n\mathrm{C}_r} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{n-2}{r-2} \dots \frac{n-r+2}{2} \times {_{n-r+1}\mathrm{C}_1}.$$

But $_{n-r+1}\mathrm{C}_1 = n-r+1$, because 1 thing can be selected from $(n-r+1)$ things in $(n-r+1)$ ways.

$$\therefore\ {_n\mathrm{C}_r} = \frac{n(n-1)(n-2)\dots(n-r+2)(n-r+1)}{r(r-1)(r-2)\dots 2.1}.$$

Multiply numerator and denominator by $(n-r)!$,

$$\therefore\ {_n\mathrm{C}_r} = \frac{n!}{r!(n-r)!}.$$

Second Method

Example 10. How many selections of 3 letters can be made from the 7 letters, a, b, c, d, e, f, g ?

Each selection of 3 letters can be arranged in 3! ways; for example, the selection *bcf* corresponds to the 6 different arrangements *bcf*, *bfc*, *cbf*, *cfb*, *fbc*, *fcb*.

But the number of arrangements of 7 things, taken 3 at a time, is $_7\mathrm{P}_3 = 7.6.5$. Therefore the number of selections is

$$_7\mathrm{P}_3 \div 3! = \frac{7.6.5}{1.2.3},$$

and we write $$_7\mathrm{C}_3 = {_7\mathrm{P}_3} \div 3! = \frac{7.6.5}{1.2.3} = \frac{7!}{3!\,4!}.$$

The number of selections, or combinations, of n unlike things, taken r at a time, is

$$ {}_nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r} = \frac{n!}{r!(n-r)!}. $$

Each selection of r unlike things can be arranged in $r!$ ways therefore each selection corresponds to $r!$ arrangements.

But the number of arrangements is

$$ {}_nP_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}. $$

Therefore the number of selections is ${}_nP_r \div r!$.

$$ \therefore\ {}_nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r} = \frac{n!}{r!(n-r)!}. $$

Since the number of selections must be an integer, the formula for ${}_nC_r$ proves that $n(n-1)(n-2)\dots(n-r+1)$ is always a multiple of $r!$, that is to say, the product of any r consecutive integers is a multiple of $r!$.

The value of ${}_nC_n$ is 1, since there is only 1 way of selecting n things from n things; thus, using the definition of 0! on p. 6, we see that the relation ${}_nC_r = \frac{n!}{r!(n-r)!}$ holds even for $r=n$. It will also hold for $r=0$, if we define ${}_nC_0$ to be 1, and this definition is also suggested by the important relation, ${}_nC_r = {}_nC_{n-r}$, which we now proceed to prove.

There are two relations which deserve special attention:

(i) ${}_nC_r = {}_nC_{n-r}$.

If r things are selected from n things, $(n-r)$ things remain. Therefore the number of selections of r things from n, namely ${}_nC_r$, is the same as the number of selections of $(n-r)$ things from n, namely ${}_nC_{n-r}$.

Alternatively, we may say

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

and

$$ {}_nC_{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}. $$

(ii) ${}_n\mathrm{C}_r + {}_n\mathrm{C}_{r-1} = {}_{n+1}\mathrm{C}_r$.

${}_{n+1}\mathrm{C}_r$ is the number of ways of selecting r letters from the $(n+1)$ unlike letters $a_1, a_2, a_3, \ldots, a_n, b$.

If each selection is made only from the letters a, the number of selections is ${}_n\mathrm{C}_r$.

If each selection contains the letter b, it contains $(r-1)$ of the n letters a, and can therefore be made in ${}_n\mathrm{C}_{r-1}$ ways.

$\therefore$ the total number of selections is ${}_n\mathrm{C}_r + {}_n\mathrm{C}_{r-1}$.

But the total number is ${}_{n+1}\mathrm{C}_r$; $\therefore$ ${}_n\mathrm{C}_r + {}_n\mathrm{C}_{r-1} = {}_{n+1}\mathrm{C}_r$.

Alternatively, we may say

$$
\begin{aligned}
{}_n\mathrm{C}_r + {}_n\mathrm{C}_{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\
&= \frac{n!\{(n-r+1)+r\}}{r!(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!} \\
&= \frac{(n+1)!}{r!(n-r+1)!} = {}_{n+1}\mathrm{C}_r.
\end{aligned}
$$

Example 11. In how many ways can a committee of 3 women and 4 men be chosen from 8 women and 7 men ? What is the number of ways if Miss X refuses to serve if Mr. Y is a member ?

(i) There are ${}_8\mathrm{C}_3$ ways of selecting 3 women from 8 women, and ${}_7\mathrm{C}_4$ ways of selecting 4 men from 7 men.

$\therefore$ the number of ways of choosing the committee is

$$
{}_8\mathrm{C}_3 \times {}_7\mathrm{C}_4 = \frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3} \times \frac{7\,.\,6\,.\,5\,.\,4}{1\,.\,2\,.\,3\,.\,4} = 1960.
$$

(ii) If both Miss X and Mr. Y are members, there remain to be selected 2 women from 7 women, and 3 men from 6 men. This can be done in ${}_7\mathrm{C}_2 \times {}_6\mathrm{C}_3 = \frac{7\,.\,6}{1\,.\,2} \times \frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3} = 420$ ways.

$\therefore$ the number of ways of making selections which do not include both Miss X and Mr. Y is $1960 - 420 = 1540$.

EXERCISE I. c

1. In how many ways can 3 books be selected from 7 different books ?

2. In how many different ways can a football eleven be chosen from 14 boys ?

3. In how many different ways can a hand of 3 cards be selected from a pack of 52 cards ?

4. Find the number of ways in which a pair of triangles can be drawn with 6 given points as vertices, no three of the points being collinear.

5. What is the greatest number of points of intersection of (i) 12 straight lines, (ii) 9 circles, (iii) 6 straight lines and 5 circles ?

6. In how many ways can a committee of 4 men and 3 ladies be formed from 10 men and 8 ladies ?

7. There are 9 houses in a row. In how many ways can 5 of the front-doors be chosen for painting green ?

8. I have written 10 letters, but only have enough stamps for 4 of them. In how many ways can I choose the letters to be stamped ?

9. In how many ways can 10 men be divided into two groups of 3 and 7 respectively ?

10. In how many ways can 2 photographs and 3 drawings be selected from 5 photographs and 5 drawings ?

11. In how many ways can a committee of 5 be chosen from 10 candidates (i) so as to include both the youngest and oldest, (ii) so as to exclude the youngest if it includes the oldest ?

12. In how many ways can 3 books be chosen from a shelf holding n different books ? In how many of these ways is the longest book included ?

13. (i) n points are marked on a circle. How many chords can be obtained by joining them in pairs ?

(ii) How many diagonals does an n-sided polygon have, if a diagonal means any line joining two non-consecutive corners ?

14. In a group of 15 boys, there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include (i) exactly 6 boy-scouts, (ii) at least 6 boy-scouts ?

15. In how many ways can a committee of 5 be chosen from 7 Conservatives and 4 Socialists so as to give a Conservative majority, if at least 1 Socialist is included ?

16. A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 men and 2 women. How many different ways can this be done if two particular women refuse to serve on the same committee ?

17. The results of 21 football matches (win, lose or draw) are to be predicted. How many different forecasts can contain exactly 18 correct results ?

18. Find n if (i) ${}_nC_3 = {}_nC_5$; (ii) ${}_nC_2 = 55$.

19. Simplify (i) ${}_{n+1}C_3 \div {}_nC_2$; (ii) ${}_{n+1}C_{r+1} \div {}_nC_r$.

20. Simplify ${}_{4n}C_{3n} \times {}_{3n}C_{2n} \times {}_{2n}C_n$.

21. Prove that ${}_nC_r = \left(\frac{n+1}{r} - 1\right) \cdot {}_nC_{r-1}$.

22. Prove that ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$.

23. Prove that the product of the first n odd numbers equals $(\frac{1}{2})^n \cdot {}_{2n}C_n \cdot {}_nP_n$.

Distribution in Groups

The number of ways of dividing $(p+q+r)$ unlike things into 3 unequal groups, containing respectively p, q, r things, is

$$\frac{(p+q+r)!}{p!\,q!\,r!}.$$

A group of p things can be selected in ${}_{p+q+r}C_p$ ways. From the remaining $(q+r)$ things, a group of q things can be selected in ${}_{q+r}C_q$ ways. This leaves r things for the third group. Therefore the total number of ways is

$$\begin{aligned} &{}_{p+q+r}C_p \times {}_{q+r}C_q \\ &= \frac{(p+q+r)!}{p!(q+r)!} \times \frac{(q+r)!}{q!\,r!} = \frac{(p+q+r)!}{p!\,q!\,r!}. \end{aligned}$$

If two or more groups contain equal numbers of things, and if no regard is paid to the order of the groups, the number of different distributions is modified. Thus if $3p$ unlike things are divided into 3 groups each containing p things, the groups can be arranged in $3!$ orders. Therefore the number of possible distributions, if no regard is paid to the order of the groups, is

$$\frac{(3p)!}{p!\,p!\,p!} \div 3!.$$

Example 12. (i) How many different hands can be held by 4 men playing bridge (13 cards each)?

(ii) How many ways can a pack of 52 cards be arranged in 4 heaps of 13 cards each?

In (i), regard must be paid to the order in which the hands are placed on the table. The number of ways is

$${}_{52}C_{13} \times {}_{39}C_{13} \times {}_{26}C_{13} = \frac{52!}{(13!)^4}.$$

In (ii), no regard is paid to the order of the heaps. Therefore the number of ways is $\frac{52!}{(13!)^4} \div 4!$.

Selections taking any Number at a Time

The number of selections from n unlike things, taking any number at a time, is $2^n - 1$.

Each thing may be selected or rejected, that is, it may be disposed of in two ways.

$\therefore$ the number of ways of disposing of the n things is

$$2 \times 2 \times 2 \times \dots n \text{ factors} = 2^n.$$

But this includes the case where all are rejected.

$\therefore$ the number of selections, if at least one thing is chosen, is $2^n - 1$.

The total number may also be obtained by considering in succession groups containing 1, 2, 3, 4, ..., n things. The number of selections is therefore

$${}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n.$$

It follows that

$${}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n = 2^n - 1.$$

A different proof of this relation is given on p. 29.

Given k unlike things and in addition p things alike of one kind, q alike of another kind, r alike of another kind, etc., the number of selections which can be made, taking any number at a time, is

$$2^k(p+1)(q+1)(r+1)\dots - 1.$$

From the p like things we can select either 0 or 1 or 2 or ... or p; we can therefore dispose of them in $(p+1)$ ways. Similarly the other groups of like things can be disposed of in $(q+1)$ ways, $(r+1)$ ways, etc.

Also each of the k unlike things can be disposed of in 2 ways (taken or left).

$\therefore$ the number of ways of disposing of them all is

$$(p+1)(q+1)(r+1)\dots(2 \times 2 \times 2 \times \dots k \text{ factors})$$
$$= 2^k(p+1)(q+1)(r+1)\dots.$$

But this includes the case where all are rejected.

$\therefore$ the number of selections, taking at least 1 thing, is

$$2^k(p+1)(q+1)(r+1)\dots - 1.$$

Example 13. How many different sums of money can be made up from five £1 notes, one 10s. note, four florins, three sixpences, one penny ?

The number is $(5+1)(1+1)(4+1)(3+1)(1+1)-1$

$$=6 \,.\, 2 \,.\, 5 \,.\, 4 \,.\, 2-1=480-1=479.$$

The value of r for which $_nC_r$ is greatest is $r=\frac{1}{2}n$ if n is even, and is $r=\frac{1}{2}(n-1)$ or $r=\frac{1}{2}(n+1)$ if n is odd.

The series of terms

$$_nC_0\,;\ _nC_1\,;\ _nC_2\,;\ _nC_3\,;\ \ldots\,;\ _nC_r\,;\ \ldots\,;\ _nC_{n-1}\,;\ _nC_n$$

can be written

$$1\,;\ \frac{n}{1}\,;\ \frac{n}{1}\cdot\frac{n-1}{2}\,;\ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\,;\ \ldots .$$

The multiplying factors which convert each term into the next, namely $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, ..., steadily decrease; and the terms themselves steadily increase so long as the multiplying factors exceed 1, and steadily decrease as soon as the multiplying factors become less than 1.

But since $_nC_r={}_nC_{n-r}$, the terms which are equidistant from the beginning and end are equal. Therefore the greatest term is the middle one when n is even, and there are two equal greatest terms, namely the two middle terms, when n is odd : these are given respectively by $r=\frac{1}{2}n$ and by $r=\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$.

EXERCISE I. d

1. In how many ways can 6 different books be divided between A, B, C so that A has 3, B has 2 and C has 1 ?

2. Two elevens are made up from 22 players. In how many arrangements will 2 particular players be on opposite sides ?

3. In how many ways can 6 people be divided into three pairs ?

4. How many selections can be made from 10 different books, if any number may be taken ?

5. How many whole numbers are factors of $2^6 . 3^4 . 5^2 . 7 . 11$, not counting 1 or the number itself ?

6. How many selections can be made from the letters, a, a, a, a, b, c, d, if any number may be taken ?

7. In how many ways can 144 be expressed as the product of 2 positive integers, counting 1×144 as one way ?

8. There are 6 points on a straight line and 8 other points. What is the greatest possible number of triangles that can ever be formed having these points as vertices ?

9. Prove that the number of ways of dividing up 20 unlike things into 5 packets of 4 each is to the number of ways of dividing them into 4 packets of 5 each as 125 is to 24.

10. 8 men are to play 4 singles at tennis, the games being simultaneous. In how many ways can the games be arranged ?

11. Two similar dice with faces numbered 1 to 6 are thrown. How many different (i) throws, (ii) totals are possible ?

12. In how many ways can a selection be made from 3 red balls, 4 blue balls, 5 green balls, if any number from 1 to 12 may be chosen ?

13. Two crews of 8 are formed from 16 oarsmen, of whom 4 can row only on bow side and 5 only on stroke side. In how many ways can this be done, regard being paid to the order in which they row ?

14. How many circles can be drawn so that each passes through 3 out of 9 given points (i) when no 4 of the points are concyclic, (ii) when 5 of the given points lie on one circle and the remaining 4 lie on another circle, no other set of 4 points being cyclic ?

15. 5 roads, A, B, C, D, E, meet at a junction, and a car comes along each road to this junction. In how many different ways may these cars continue their journeys across the junction ?
In how many of these cases do exactly 2 of the cars go down the road A ?

16. If ${}_{3n}C_r = {}_{3n}C_{n+r}$, express the value of each in terms of n.

17. Prove that ${}_nC_r + 2 \cdot {}_nC_{r-1} + {}_nC_{r-2} = {}_{n+2}C_r$.

18. Prove that the greatest value of ${}_{2n}C_r$ is double the greatest value of ${}_{2n-1}C_r$.

19. In how many ways can $2n$ people be divided into n couples?

20. Find the number of ways in which 6 men and 6 women can be arranged in 3 sets for tennis (i) if there is no restriction, (ii) if each man has a woman as partner.

21. There are $2n$ things of which n are alike and the rest are unlike. How many selections can be made, if any number may be taken ?

22. There are m points on one straight line AB and n points on another line AC, none of them being the point A. How many triangles can be formed with these points as vertices ? How many, if the point A is added ?

MISCELLANEOUS EXAMPLES

EXERCISE I. e

1. In how many ways can 9 different books be arranged on a shelf if two particular books are separated ?

2. In how many ways can n boys sit in a row if two particular boys are excluded from being at either end, and if two other boys must sit next to each other ?

3. In how many ways can three cards be selected from a pack of 52 cards, if at least one of them is to be an ace ?

4. Four letters are written and four envelopes addressed. In how many ways can all the letters be placed in the wrong envelopes ?

5. In how many ways can two sets of tennis, 4 players in each set, be arranged if 10 people are available ?

6. How many even numbers greater than 300 can be formed with the digits 1, 2, 3, 4, 5, no repetitions being allowed ?

7. Find the number of arrangements of letters in *alchemist* if the order of the consonants must not be changed.

8. Prove that there are about 40 million different arrangements of the letters in *inequalities*.

9. In how many ways can a committee of 2 Englishmen, 2 Frenchmen, 1 American be chosen from 6 Englishmen, 7 Frenchmen, 3 Americans ? In how many of these ways do a particular Englishman and a particular Frenchman belong to the committee ?

10. Attempts are made to predict the results (win, draw, lose) of 10 football matches. In how many different ways can exactly 6 correct results be given ?

11. Find the number of arrangements of the letters in *ratatatatat*. How many are there in which the *a*'s are all separated from one another ?

12. A detachment consists of 3 sergeants, 4 corporals and 70 privates. In how many ways can a guard of 6 be selected to include 1 sergeant, 1 corporal and 4 privates ?

13. In how many ways can a lawn-tennis mixed double be made up from 5 married couples, if no husband and wife play in the same set ?

14. $2m$ white counters and $2n$ red counters are arranged in a straight line with $(m+n)$ counters on each side of a central mark. Find how many of the arrangements are symmetrical with respect to this mark.

15. Find the number of ways in which n boys can be arranged in a line so that 3 particular boys are separated.

16. Prove that $(2n+1)(2n+3)(2n+5)\ldots(4n-3)(4n-1)$ equals $\left(\frac{1}{2}\right)^n \frac{(4n)!\,n!}{(2n)!\,(2n)!}$.

17. In how many ways can 9 different books be labelled, so that 4 of them have red labels, 3 have blue labels, and 2 have green labels?

18. What is the greatest number of points of intersection made by m straight lines and n circles?

19. There are 3 pigeon holes marked A, B, C. In how many ways can I arrange 10 different postcards so that 5 of them are in A, 3 in B, and 2 in C?

20. A network is formed by one set of p parallel lines and another set of q parallel lines. How many parallelograms are there in the net?

21. If ABCDEF are 6 points *in any order* on a circle, the points of intersection of AB, DE; BC, EF; CD, FA lie on a straight line called the *Pascal* line of the hexagon ABCDEF. How many Pascal lines are determined by 6 points on a circle?

22. In how many ways can p boys and q girls be arranged in a row, if the boys are put in the order of their ages, but not necessarily together, with the oldest boy at the left end of the row?

23. In how many ways can r things be selected from n unlike things, if two particular things must not occur in the same selection?

24. Explain why ${}_nC_r + {}_{n-1}C_r + {}_{n-2}C_r + \ldots + {}_rC_r$ equals ${}_{n+1}C_{r+1}$.

25. Find the number of selections of n things from $2n$ things of which n are alike and the rest are unlike.

26. In how many ways can the $2n$ letters $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, be arranged in a line so that the suffixes of the letters x and also those of the letters y are respectively in ascending order of magnitude?

27. (i) Find the number of ways of putting p noughts and q crosses in a line, $p > q$, if no two crosses are together and if a nought comes at each end.

(ii) Find the number of ways in which p like jobs may be assigned to $(q+1)$ men, if no man is left unemployed, $p > q$, by showing this problem is equivalent to (i).

28. Prove that, if n and p are positive integers, $(np)!$ is a multiple of $(p!)^n . n!$.

CHAPTER II

BINOMIAL THEOREM: POSITIVE INTEGRAL INDEX

Product of Binomial Factors

The expansion of $(x+a_1)(x+a_2)$ is the sum of the terms obtained by multiplying together 2 letters, one taken from each of the 2 factors.

$$(x+a_1)(x+a_2)=x^2+xa_1+xa_2+a_1a_2.$$

It is convenient to group them according to the number of x factors in the terms.

$$(x+a_1)(x+a_2)=x^2+x(a_1+a_2)+a_1a_2.$$

Similarly,

$$(x+a_1)(x+a_2)(x+a_3)$$
$$=x^3+x^2(a_1+a_2+a_3)+x(a_2a_3+a_3a_1+a_1a_2)+a_1a_2a_3.$$

Similarly,

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)$$
$$=x^4+x^3(a_1+a_2+a_3+a_4)$$
$$+x^2(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)$$
$$+x(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)+a_1a_2a_3a_4.$$

This may be written

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)=x^4+s_1x^3+s_2x^2+s_3x+s_4,$$

where $s_1=$ the sum of the terms a_1, a_2, a_3, a_4,

$s_2=$ the sum of their products, two at a time,

$s_3=$ the sum of their products, three at a time,

$s_4=$ the product, four at a time, $a_1a_2a_3a_4$.

This process can be applied to as many factors as required.

Thus $$(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)$$
$$=x^n+s_1x^{n-1}+s_2x^{n-2}+s_3x^{n-3}+\ldots+s_{n-1}x+s_n,$$

where s_1 contains n terms, $(a_1+a_2+\ldots+a_n)$,

s_2 contains ${}_nC_2$ terms, $(a_1a_2+a_1a_3+a_2a_3+\ldots)$,

because it consists of the selections, two at a time, from n unlike things,

$$s_3 \text{ contains } {}_nC_3 \text{ terms, } (a_1a_2a_3 + a_1a_2a_4 + \dots),$$

and, in general, s_r contains ${}_nC_r$ terms.

If $a_1 = a_2 = a_3 = \dots = a_n = a$, the left side becomes $(x+a)^n$.

Also
$$s_1 = a + a + \dots, \quad n \text{ terms, } = na;$$
$$s_2 = a^2 + a^2 + \dots, \quad {}_nC_2 \text{ terms, } = {}_nC_2a^2;$$
$$s_3 = a^3 + a^3 + \dots, \quad {}_nC_3 \text{ terms, } = {}_nC_3a^3;$$

and so on.

Therefore, **if n is any positive integer,**

$$\begin{aligned}(\mathbf{x}+\mathbf{a})^{\mathbf{n}} &= \mathbf{x}^{\mathbf{n}} + \mathbf{n}\mathbf{x}^{\mathbf{n}-1}\mathbf{a} + {}_{\mathbf{n}}\mathbf{C}_2\mathbf{x}^{\mathbf{n}-2}\mathbf{a}^2 + \dots + {}_{\mathbf{n}}\mathbf{C}_{\mathbf{r}}\mathbf{x}^{\mathbf{n}-\mathbf{r}}\mathbf{a}^{\mathbf{r}} + \dots + \mathbf{a}^{\mathbf{n}} \\ &= \mathbf{x}^{\mathbf{n}} + \mathbf{n}\mathbf{x}^{\mathbf{n}-1}\mathbf{a} + \frac{\mathbf{n}(\mathbf{n}-1)}{1.2}\mathbf{x}^{\mathbf{n}-2}\mathbf{a}^2 + \frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)}{1.2.3}\mathbf{x}^{\mathbf{n}-3}\mathbf{a}^3 + \dots \\ &\quad + \frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)\dots(\mathbf{n}-\mathbf{r}+1)}{1.2.3\dots\mathbf{r}}\mathbf{x}^{\mathbf{n}-\mathbf{r}}\mathbf{a}^{\mathbf{r}} + \dots + \mathbf{a}^{\mathbf{n}}.\end{aligned}$$

This is called the **Binomial Theorem** for a positive integral index.

It may also be proved by "induction." The method of induction consists in showing that, *if* the theorem is true for some special integral value of n, say $n = k$, then it is true for $n = k+1$. Consequently, if it is true for $n = 2$, then it is true for $n = 3$, and therefore for $n = 4$, and so on indefinitely.

Suppose that the theorem is true for $n = k$, that is, suppose that

$$\begin{aligned}(x+a)^k = x^k &+ {}_kC_1x^{k-1}a + {}_kC_2x^{k-2}a^2 + \dots \\ &+ {}_kC_{r-1}x^{k-r+1}a^{r-1} + {}_kC_rx^{k-r}a^r + \dots + a^k.\end{aligned}$$

Then
$$\begin{aligned}(x+a)^{k+1} &= (x+a)(x+a)^k \\ &= x^{k+1} + (1 + {}_kC_1)x^ka + ({}_kC_1 + {}_kC_2)x^{k-1}a^2 + \dots \\ &\quad + ({}_kC_{r-1} + {}_kC_r)x^{k-r+1}a^r + \dots + a^{k+1}.\end{aligned}$$

But ${}_kC_{r-1} + {}_kC_r = {}_{k+1}C_r$, see p. 13,

$$\begin{aligned}\therefore\ (x+a)^{k+1} = x^{k+1} &+ {}_{k+1}C_1x^ka + {}_{k+1}C_2x^{k-1}a^2 + \dots \\ &+ {}_{k+1}C_rx^{k-r+1}a^r + \dots + a^{k+1}.\end{aligned}$$

$\therefore$ *if* the theorem is true for $n = k$, it is true for $n = k+1$.

But it is true when $n = 2$ because

$$(x+a)^2 = x^2 + 2xa + a^2 = x^2 + {}_2C_1xa + a^2,$$

$\therefore$ it is true when $n = 3$, and $\therefore$ when $n = 4$, and so on indefinitely.

$\therefore$ it is true for all positive integral values of n.

Properties of the Binomial Expansion

(i) The expansion of $(x+a)^n$ contains $(n+1)$ terms, of which ${}_nC_r x^{n-r}a^r$ is the **(r + 1)th term**; this term or the rth term is called the *general term*.

Also the coefficients of terms equidistant from the two ends are equal, for the $(r+1)$th term from the end is ${}_nC_{n-r}x^r a^{n-r}$, and ${}_nC_r = {}_nC_{n-r}$, see p. 12.

(ii) If n is even, there is one middle term given by putting $r=\frac{1}{2}n$ in ${}_nC_r x^{n-r}a^r$. If n is odd, there are two middle terms given by putting $r=\frac{1}{2}(n-1)$ and $r=\frac{1}{2}(n+1)$.

(iii) If $a=1$, we have

$$(1+x)^n = 1 + {}_nC_1 x + {}_nC_2 x^2 + \ldots + {}_nC_r x^r + \ldots + x^n.$$

(iv) If we write $-a$ for a, we have

$$(x-a)^n = x^n - {}_nC_1 x^{n-1}a + {}_nC_2 x^{n-2}a^2 - \ldots + (-1)^r {}_nC_r x^{n-r}a^r + \ldots + (-1)^n a^n.$$

Some practice in writing down binomial expansions has been given in Part III of *A New Algebra*, Ch. VIII, p. 123. For convenience, further examples are added here.

EXERCISE II. a

[*Large coefficients should be left in factors, not multiplied out.*]

1. Write down the expansions of

(i) $(x+a)(x+b)(x+c)$; (ii) $(x+2a)(x+2b)(x+2c)$;

(iii) $(x+a)(x+b)(x+c)(x+d)$.

2. What is the coefficient of x^3 in the expansion of

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)\ ?$$

How many terms are there in the coefficient of x^2? What are the coefficients of x^3 and x^2 in $(x+a)^5$?

3. How many terms are there in the coefficients of x^{n-2}, x^2, x^r in the expansion of

$$(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)\ ?$$

What are the coefficients of x^{n-2}, x^2, x^r in $(x+a)^n$?

Write down the expansions of Nos. 4-11:

4. $(x+1)^4$. **5.** $(y-1)^4$. **6.** $(x+2a)^3$. **7.** $(z+2)^4$.

8. $(a-b)^5$. **9.** $\left(y+\frac{1}{y}\right)^6$. **10.** $(3x-2y)^4$. **11.** $(x^2+3y^2)^5$.

12. For the expansion of $(x+1)^8$, find (i) the number of terms, (ii) which term involves x^4, (iii) the power of x in the 7th term, (iv) the coefficient of x^5.

13. For the expansion of $(5x-3y)^{20}$, find (i) the number of terms, (ii) the powers of x and y in the 6th term and in the 12th term, (iii) the 3rd term, the 19th term, the middle term.

14. Find the coefficient of x^2 in the expansions of

(i) $(1+2x)^9$; (ii) $(1-3x)^7$; (iii) $(x+3)^6$; (iv) $(2x-5)^8$.

15. Find the coefficient of x^3 in the expansions of

(i) $(x-10)^5$; (ii) $(2x-\frac{1}{2})^6$; (iii) $(x-y)^{10}$; (iv) $\left(x+\dfrac{1}{x}\right)^7$.

16. Find the 4th term in the expansions of

(i) $(2a-5b)^7$; (ii) $(1+3x)^n$; (iii) $\left(x+\dfrac{1}{x}\right)^n$.

17. Find the $(r+1)$th term in the expansions of

(i) $(x+y)^n$; (ii) $(a-2b)^n$; (iii) $\left(x+\dfrac{1}{x}\right)^n$.

18. Expand in ascending powers of x:

(i) $(1+x)^n+(1-x)^n$; (ii) $(1+x)^n-(1-x)^n$.

19. Evaluate $(1{\cdot}0025)^{10}$ correct to 6 decimal places.

20. Use the binomial expansion to show that £1000 will amount to £1480, to the nearest £, in 10 years at 4 per cent. per annum compound interest.

21. What is the coefficient of x^{20} in $(x^2+3x)^{12}$?

22. What is the coefficient of x^n in $(1+\frac{1}{4}x)^{2n}$?

23. (i) What is the middle term of the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$?

(ii) What are the two middle terms for $\left(x-\dfrac{1}{x}\right)^{2n+1}$?

24. Simplify $(a+b)^3-3b(a+b)^2+3b^2(a+b)-b^3$.

25. Simplify $(x-1)^4+4(x-1)^3+6(x-1)^2+4(x-1)+1$.

26. Find the coefficient of x^4 in the expansions of

(i) $(1-x)(1+x)^5$; (ii) $(1+x)(1-x)^n$.

27. Find the first 4 terms in the expansion in ascending powers of x of $(1+2x)(1-x^3)^9$. What is the coefficient of x^{19}?

28. Simplify $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$.

29. Find (i) the sum, (ii) the product of $(2+\sqrt{3})^7$ and $(2-\sqrt{3})^7$. Hence show that the integral part of $(2+\sqrt{3})^7$ is 10,083.

30. Simplify $x^n(x-1)^n+{}_nC_1x^{n-1}(x-1)^{n-1}(x+1)$
$+\ldots+{}_nC_rx^{n-r}(x-1)^{n-r}(x+1)^r+\ldots(x+1)^n$.

Special Terms

The following examples illustrate the selection of particular terms from an expansion.

Example 1. Find the term independent of x in the expansion* of $\left(3x - \frac{5}{x^3}\right)^8$.

The general term is ${}_8C_r(3x)^{8-r}\left(-\frac{5}{x^3}\right)^r$ in which the power of x is $8-4r$. Taking $r=2$, we get ${}_8C_2(3x)^6\left(-\frac{5}{x^3}\right)^2$, which equals

$$\frac{8.7}{1.2}.3^6.5^2=700\times 3^6.$$

Alternatively, as follows :

$$\left(3x-\frac{5}{x^3}\right)^8=\left(\frac{3x^4-5}{x^3}\right)^8=\frac{(3x^4-5)^8}{x^{24}}.$$

$\therefore$ we require the coefficient of x^{24} in $(3x^4-5)^8$.

The term in x^{24} is ${}_8C_2(3x^4)^6(-5)^2$;

$$\therefore \text{ the coefficient is } \frac{8.7}{1.2}.3^6.5^2=700\times 3^6.$$

Example 2. Expand $(2+3x-x^2)^n$ in ascending powers of x as far as x^2.

$$\begin{aligned}(2+3x-x^2)^n &= [2+x(3-x)]^n\\ &= 2^n+n.2^{n-1}x(3-x)+\frac{n(n-1)}{1.2}.2^{n-2}x^2(3-x)^2+\dots\\ &= 2^n+3nx.2^{n-1}-nx^2.2^{n-1}+(n^2-n).2^{n-3}.9x^2+\dots\\ &= 2^n+3nx.2^{n-1}+(9n^2-13n)x^2.2^{n-3}+\dots.\end{aligned}$$

Greatest Coefficients and Terms

Since the coefficient of x^r in the expansion of $(1+x)^n$ is ${}_nC_r$, it follows from p. 17 that, if n is even, the coefficient is greatest when $r=\frac{1}{2}n$, and that, if n is odd, there are two greatest coefficients given by $r=\frac{1}{2}(n-1)$ and $r=\frac{1}{2}(n+1)$. Thus if n is even the middle term has the greatest coefficient, and, if n is odd, the two middle terms have equal coefficients which are greater than all the others.

To find the greatest term in the expansion of $(1+x)^n$, where x is positive, and n is a positive integer.

Denote the rth term by u_r.

Then
$$u_{r+1}=\frac{n(n-1)(n-2)\dots(n-r+2)(n-r+1)}{r!}x^r$$
$$=\frac{n(n-1)(n-2)\dots(n-r+2)}{(r-1)!}x^{r-1}\times\frac{n-r+1}{r}x$$
$$=u_r\times\frac{(n-r+1)}{r}x.$$

$\therefore\ u_{r+1}>u_r$ if $(n-r+1)x>r$ or $nx-rx+x>r$,

or if $r+rx<nx+x$ or if $r(1+x)<(n+1)x$.

$$\therefore\ u_{r+1}>u_r \text{ whenever } r<\frac{(n+1)x}{1+x}.$$

Similarly, $$u_{r+1}<u_r \text{ whenever } r>\frac{(n+1)x}{1+x}.$$

(i) If $\dfrac{(n+1)x}{1+x}=$ an integer, q say, then $u_{q+1}=u_q$ and u_q, u_{q+1} are greater than any of the other terms.

(ii) If $\dfrac{(n+1)x}{1+x}=q+$ a positive fraction less than 1, then $u_{r+1}>u_r$ whenever $r<q$ and also if $r=q$; but $u_{r+1}<u_r$ if $r=q+1$ and whenever $r>q+1$.

$$\therefore\ u_{q+1}>u_q>u_{q-1}\dots \quad\text{and}\quad u_{q+1}>u_{q+2}>u_{q+3}\dots.$$

$$\therefore\ u_{q+1} \text{ is the greatest term.}$$

The greatest term in the expansion of $(a+x)^n$, where a, x are positive and n is a positive integer, may be deduced in the same way from the corresponding relation,

$$u_{r+1}=u_r\times\frac{(n-r+1)}{r}\cdot\frac{x}{a}.$$

If a or x is negative, this method gives the *numerically* greatest term.

Example 3. Find which is the numerically greatest term in the expansion of $(5-4x)^{12}$ when $x=\frac{2}{3}$.

The terms in the expansion are alternately positive and negative, but the *numerically* greatest term will be the same term as in the expansion of $(5+4x)^{12}$.

If u_r denotes the rth term, $u_{r+1} > u_r$ if $\dfrac{12-r+1}{r} \cdot \dfrac{4x}{5} > 1$ or if $8(13-r) > 15r$ since $x = \frac{2}{3}$.

$$\therefore\ u_{r+1} > u_r \text{ if } 104 - 8r > 15r \text{ or } 23r < 104 \text{ or } r < 4\tfrac{12}{23}.$$

$$\therefore\ u_5 > u_4 > u_3 \ldots \quad \text{and} \quad u_5 > u_6 > u_7 \ldots.$$

$\therefore$ the 5th term is the greatest.

EXERCISE II. b

[Large coefficients should be left in factors, not multiplied out.]

1. Write down the named terms in the following expansions :

(i) 4th term in $(2x-3y)^7$; (ii) 10th term in $\left(3x+\dfrac{2}{x}\right)^{12}$;

(iii) 6th term in $\left(\dfrac{x^2}{2}-\dfrac{x}{3}\right)^8$; (iv) rth term in $\left(x+\dfrac{1}{x}\right)^n$.

2. Write down the coefficients of the named terms in the following expansions :

(i) x^8 in $(2-x)^{11}$; (ii) x^9 in $\left(2x-\dfrac{3}{x}\right)^{13}$;

(iii) x^{11} in $(3x+2x^2)^9$; (iv) x^{2r} in $(1-x^2)^{4r}$;

(v) x^{r+1} in $(1-x)^{n-1}$; (v) x^{2r} in $\left(x+\dfrac{1}{x}\right)^{4r}$.

3. Write down the terms independent of x in the following expansions :

(i) $\left(x-\dfrac{2}{x}\right)^{10}$; (ii) $\left(2x^3-\dfrac{1}{x}\right)^{12}$; (iii) $\left(x+\dfrac{1}{x}\right)^{2n}$.

4. Write down the coefficients of the named terms in the following expansions :

(i) x^2 in $\left(2x^3-\dfrac{1}{2x}\right)^{18}$; (ii) x^3 in $\left(x^2-\dfrac{1}{2x}\right)^{18}$;

(iii) x^7 in $\left(2x-\dfrac{3}{x}\right)^{15}$; (iv) x^7 in $\left(2x^2-\dfrac{1}{4x}\right)^{11}$.

5. Expand $(1-2x+3x^2)^7$ in ascending powers of x as far as x^3.

6. Find the coefficient of x^4 in the expansion of $(2+x-x^2)^4$.

7. Find the coefficient of x^7 in the expansion of $(1-x^2-x^3)^n$.

8. What is the ratio of the $(r+1)$th term to the rth term in the expansions of

(i) $\left(1+\dfrac{x}{2}\right)^n$; (ii) $\left(x-\dfrac{3}{x}\right)^n$; (iii) $(2x+3y)^{8r}$?

9. What is the ratio of the $(r+1)$th term to the $(r-1)$th term in the expansions of (i) $(x-y)^n$; (ii) $(2x+3y)^n$?

10. Find the value of r if the coefficients of x^r and of x^{r+1} in $(3x+2)^{19}$ are equal.

11. If $x=0{\cdot}2$, prove that the 11th term in the expansion of $(1+x)^{14}$ is $\frac{1}{10}$th of the 10th term.

12. What is the greatest coefficient in the expansions of

(i) $(1+x)^{10}$; (ii) $(1+x)^{11}$; (iii) $(1+x)^{4n+3}$?

13. In the expansion of $(1+x)^{12}$, the ratio of a certain coefficient to the preceding coefficient is $\frac{6}{7}$; find the numerical values of the two coefficients.

14. Find which term in the expansion of $\left(x+\dfrac{1}{2x}\right)^{3n}$ has the largest coefficient.

15. Find which are the numerically greatest terms in the expansions of the following:

(i) $(1+2x)^9$ if $x=\frac{1}{3}$; (ii) $(1+2x)^9$ if $x=3$;

(iii) $(7+x)^{29}$ if $x=3$; (iv) $(1+3x)^7$ if $x=\frac{1}{4}$;

(v) $(ax-by)^{10}$ if $a=2$, $b=5$, $x=3$, $y=\frac{1}{2}$?

16. The expression $(3+5)^{20}$ is expanded by the binomial theorem. Prove that each of the 2nd, 3rd, ..., 10th terms is more than double the term which precedes it, and that each of the 11th, 12th, ..., 21st terms is less than double the term which precedes it.

17. Prove that the coefficients of x^2 and x^3 in the expansion of $(x^2+2x+2)^n$ are $2^{n-1}.n^2$ and $\frac{1}{3}n(n^2-1).2^{n-1}$.

18. Find the term independent of x in the expansion of

$$\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^5.$$

19. Expand $\left(x+\dfrac{1}{x}\right)^4\left(x-\dfrac{1}{x}\right)^3$ in descending powers of x.

20. Expand $(1+x+x^2)^n$ in ascending powers of x as far as x^3.

21. If a_r is the coefficient of x^r in $(1+bx^2+cx^3)^n$, prove that $2na_4=(n-1)a_2{}^2$.

22. Find the coefficient of x^r in the expansion of

(i) $(1-x)(1+x)^n$; (ii) $(1+2x+x^2)(1+x)^n$.

23. Prove that

$$(1+x)^{2n}-2nx(1+x)^{2n-1}+\frac{2n(2n-2)}{2!}x^2(1+x)^{2n-2}$$
$$-\frac{2n(2n-2)(2n-4)}{3!}x^3(1+x)^{2n-3}+\ldots(n+1)\text{ terms}$$

equals $(1-x^2)^n$.

24. If n is a positive integer, prove that

$$\left\{1+n+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}+\ldots\right\}\times$$
$$\left\{1+2n+\frac{2n(2n-1)}{2!}+\frac{2n(2n-1)(2n-2)}{3!}+\ldots\right\}.$$

equals $$1+3n+\frac{3n(3n-1)}{2!}+\frac{3n(3n-1)(3n-2)}{3!}+\ldots.$$

25. (i) If $(x_1+y_1+z_1)(x_2+y_2+z_2)\ldots(x_n+y_n+z_n)$ is expanded, find how many terms contain p factors x, q factors y, r factors z, where $p+q+r=n$.

Hence find the coefficient of $x^p y^q z^r$ in the expansion of

$$(x+y+z)^n.$$

(ii) What are the coefficients of x^3y, x^2y^2, x^2yz, $xyzw$ in

$$(x+y+z+w)^4\ ?$$

26. Find the coefficient of $\frac{1}{x}$ in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.

27. Find the coefficient of x^r in the expansion of

$$(x+2)^n+(x+2)^{n-1}(x+1)+(x+2)^{n-2}(x+1)^2+\ldots+(x+1)^n.$$

28. If $(1+x^2)^2(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+\ldots$, and if a_0, a_1, a_2 are in A.P., show that there are two possible values of n and find them.

29. If the middle term of the expansion of $(1+x)^{2n}$ is the greatest term, prove that x lies between $1-\frac{1}{n+1}$ and $1+\frac{1}{n}$.

Relations between Binomial Coefficients

For the rest of the chapter, we shall for simplicity denote ${}_nC_r$ by c_r.

Thus $\quad (1+x)^n \equiv c_0+c_1x+c_2x^2+\ldots+c_rx^r+\ldots+c_nx^n.$

Here, $c_0, c_1, c_2, \ldots, c_n$ are called *binomial coefficients.*

In the expansion of $(1+x)^n$,

(i) the sum of the coefficients is 2^n,

(ii) the sum of the coefficients of the odd terms equals that of the even terms, each being 2^{n-1}.

(i) In the expansion of $(1+x)^n$, put $x=1$;

$$\therefore\ (1+1)^n=c_0+c_1+c_2+\ldots+c_n;$$
$$\therefore\ \mathbf{c_0+c_1+c_2+\ldots+c_n=2^n.}$$

(ii) In the expansion of $(1+x)^n$, put $x=-1$;

$$\therefore\ (1-1)^n=c_0-c_1+c_2-\ldots+(-1)^n c_n;$$

$$\therefore\ c_0+c_2+c_4+\ldots=c_1+c_3+c_5+\ldots.$$

But the sum of all the coefficients is 2^n,

$$\therefore\ c_0+c_2+c_4+\ldots=(\tfrac{1}{2}\text{ of }2^n)=2^{n-1}.$$

Note. An alternative proof of (i) was given on p. 16.

Finite Series involving Binomial Coefficients

The most useful methods of summation may be classified as follows:

(i) Express the series as the sum of two or more binomial expansions.

(ii) Obtain the series by differentiation or integration of a finite series whose sum is known.

(iii) Build up a function in which the given series is the coefficient of a particular power of the variable and evaluate this coefficient in an independent manner.

The following examples illustrate these methods.

Example 4. Sum the series

$$c_0+2c_1x+3c_2x^2+\ldots+(n+1)c_nx^n.$$

First Method

The series $=(c_0+c_1x+c_2x^2+\ldots+c_nx^n)+(c_1x+2c_2x^2+\ldots+nc_nx^n)$.

The first bracket $=(1+x)^n$;

the second bracket

$$=nx+2.\frac{n(n-1)}{2!}x^2+3.\frac{n(n-1)(n-2)}{3!}x^3+\ldots+nx^n$$

$$=nx\left\{1+(n-1)x+\frac{(n-1)(n-2)}{2!}x^2+\ldots+x^{n-1}\right\}$$

$$=nx(1+x)^{n-1}.$$

$$\therefore\ \text{the series}=(1+x)^n+nx(1+x)^{n-1}.$$

Second Method

$$c_0x+c_1x^2+c_2x^3+\ldots+c_nx^{n+1}\equiv x(1+x)^n.$$

Differentiate each side w.r.t. x;

$$\therefore\ c_0+2c_1x+3c_2x^2+\ldots+(n+1)c_nx^n=(1+x)^n+nx(1+x)^{n-1}.$$

Note. The sum of the series

$$c_0 + 2c_1 + 3c_2 + \dots + (n+1)c_n$$

is obtained by putting $x = 1$.

The sum $= (1+1)^n + n(1+1)^{n-1} = 2^n + n \cdot 2^{n-1} = (n+2) \cdot 2^{n-1}$.

Example 5. Find the value of

(i) $c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2$; (ii) $c_0c_1 + c_1c_2 + c_2c_3 + \dots + c_{n-1}c_n$.

$$(c_0 + c_1x + c_2x^2 + \dots + c_nx^n)(c_0 + c_1x + c_2x^2 + \dots + c_nx^n) \equiv (1+x)^n(1+x)^n.$$

(i) The coefficient of x^n on the left side

$$= c_0c_n + c_1c_{n-1} + c_2c_{n-2} + \dots + c_nc_0$$
$$= c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2, \text{ since } c_r = c_{n-r}.$$

$\therefore$ $c_0^2 + c_1^2 + \dots + c_n^2$ equals the coefficient of x^n in $(1+x)^{2n}$, and this is ${}_{2n}\mathrm{C}_n = \dfrac{(2n)!}{n!\,n!}$.

$$\therefore\ c_0^2 + c_1^2 + \dots + c_n^2 = \frac{(2n)!}{n!\,n!}.$$

(ii) The coefficient of x^{n-1} on the left side

$$= c_0c_{n-1} + c_1c_{n-2} + c_2c_{n-3} + \dots + c_{n-1}c_0$$
$$= c_0c_1 + c_1c_2 + c_2c_3 + \dots + c_{n-1}c_n, \text{ since } c_r = c_{n-r}$$
$$= \text{coefficient of } x^{n-1} \text{ in } (1+x)^{2n}$$
$$= {}_{2n}\mathrm{C}_{n-1} = \frac{(2n)!}{(n-1)!\,(n+1)!}.$$

Example 6. Sum the series

$$c_1 + 2^2c_2x + 3^2c_3x^2 + 4^2c_4x^3 + \dots + n^2c_nx^{n-1}$$

and deduce the value, if $n > 2$, of

$$c_1 - 2^2c_2 + 3^2c_3 - \dots + (-1)^{n-1}n^2c_n.$$

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \dots + c_nx^n \equiv (1+x)^n.$$

Differentiate w.r.t. x,

$$\therefore\ c_1 + 2c_2x + 3c_3x^2 + \dots + nc_nx^{n-1} \equiv n(1+x)^{n-1};$$
$$\therefore\ c_1x + 2c_2x^2 + 3c_3x^3 + \dots + nc_nx^n \equiv nx(1+x)^{n-1}.$$

Differentiate w.r.t. x,

$$\therefore\ c_1 + 2^2c_2x + 3^2c_3x^2 + \dots + n^2c_nx^{n-1}$$
$$\equiv n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2};$$
$$\therefore\ \text{the sum} = n(1+x)^{n-2}(1+nx).$$

If $x=-1$, $(1+x)^{n-2}=0$ for $n>2$;

$$\therefore\ c_1-2^2c_2+3^2c_3-\dots+(-1)^{n-1}n^2c_n=0.$$

Example 7. Prove that

$$c_1-\tfrac{1}{2}c_2+\tfrac{1}{3}c_3-\dots+(-1)^{n-1}\frac{1}{n}c_n=1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{n}.$$

$$1+c_1x+c_2x^2+c_3x^3+\dots+c_nx^n\equiv(1+x)^n;$$

$$\therefore\ c_1+c_2x+c_3x^2+\dots+c_nx^{n-1}\equiv\frac{(1+x)^n-1}{x}.$$

Integrate from $x=-1$ to $x=0$.

$$\therefore\ c_1-\tfrac{1}{2}c_2+\tfrac{1}{3}c_3+\dots+(-1)^{n-1}\frac{1}{n}c_n$$

$$=\int_{-1}^{0}\frac{(1+x)^n-1}{x}\,dx=\int_0^1\frac{y^n-1}{y-1}\,dy,\text{ where } 1+x=y,$$

$$=\int_0^1(1+y+y^2+\dots+y^{n-1})dy$$

$$=1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{n}.$$

EXERCISE II. c

1. What relations are obtained by putting (i) $x=2$, (ii) $x=-2$ in the expansion of $(1+x)^n$?

2. What is the sum of the coefficients of the powers of x, including x^0, in the expansion of (i) $(1+x)^8$; (ii) $(1+x+x^2)^6$?

3. If $(1+2x+2x^2)^3\equiv a_0+a_1x+a_2x^2+\dots+a_6x^6$, find the values of (i) $a_0+a_1+a_2+\dots+a_6$; (ii) $a_0+a_2+a_4+a_6$; (iii) $a_1+a_3+a_5$.

4. Expand $(1-2x+3x^2)^5$ in ascending powers of x as far as x^4. Find in the complete expansion (i) the algebraic sum of the coefficients of all the powers of x, (ii) the sum of all these coefficients, each taken positively, (iii) the algebraic sum of the coefficients of all the odd powers of x.

5. Prove that $(1+c_1+c_2+\dots+c_n)^2$ equals
$$1+{}_{2n}C_1+{}_{2n}C_2+\dots+{}_{2n}C_{2n}.$$

6. Obtain relations by equating the coefficients of x^r in the expansions of each side of the following identities:
(i) $(1+x)^{n+1}\equiv(1+x)(c_0+c_1x+\dots+c_nx^n)$;
(ii) $(1+x)^{n+2}\equiv(1+2x+x^2)(c_0+c_1x+\dots+c_nx^n)$;
(iii) $(1+x)^{n+3}\equiv(1+3x+3x^2+x^3)(1+x)^n$.

7. Prove that (i) $c_1+2c_2+3c_3+\dots+nc_n=n\,.\,2^{n-1}$.
(ii) $c_1-2c_2+3c_3+\dots+(-1)^{n-1}nc_n=0$.

8. What is the value of $c_0 - 2c_1 + 3c_2 - \dots + (-1)^n (n+1)c_n$?

9. Sum the series $c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \dots + c_n \frac{x^{n+1}}{n+1}$.

Prove that $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \dots + \frac{1}{n+1}c_n = \frac{1}{n+1}(2^{n+1} - 1)$.

[For a calculus method, use integration.]

10. Sum the series

(i) $c_0 + c_2 x^2 + c_4 x^4 + \dots + c_n x^n$, n even ;

(ii) $c_0 + c_2 x^2 + c_4 x^4 + \dots + c_{n-1} x^{n-1}$, n odd.

11. Use the identity $(1+x)^n (1-x)^n \equiv (1-x^2)^n$ to show that

$$c_0^2 - c_1^2 + c_2^2 - \dots + (-1)^n c_n^2 = (-1)^{\frac{1}{2}n} \frac{n!}{(\frac{1}{2}n)!\,(\frac{1}{2}n)!}$$

if n is even.

What is its value if n is odd ?

12. Prove that $c_0 c_r - c_1 c_{r-1} + c_2 c_{r-2} - \dots + (-1)^r c_r c_0$ equals

$$(-1)^{\frac{1}{2}r} \frac{n!}{(\frac{1}{2}r)!\,(n - \frac{1}{2}r)!} \text{ if } r \text{ is even.}$$

What is its value if r is odd ?

13. Use the identity $(1+x)^m (1+x)^n = (1+x)^{m+n}$ to prove *Vandermonde's* theorem,

$${}_m C_r + {}_m C_{r-1}\, {}_n C_1 + {}_m C_{r-2}\, {}_n C_2 + \dots + {}_n C_r = {}_{m+n} C_r.$$

14. Prove that

$$c_1 + 3c_3 + 5c_5 + \dots = 2c_2 + 4c_4 + 6c_6 + \dots = n \,.\, 2^{n-2}.$$

15. Prove that $c_0 + 3c_1 + 5c_2 + \dots + (2n+1)c_n = (n+1) \,.\, 2^n$.

16. Prove that $c_2 + 2c_3 + 3c_4 + \dots + (n-1)c_n = 1 + (n-2) \,.\, 2^{n-1}$.

17. Prove that $c_1^2 + 2c_2^2 + 3c_3^2 + \dots + nc_n^2 = \frac{(2n-1)!}{(n-1)!\,(n-1)!}$.

18. Prove that

$$c_0^2 + 2c_1^2 + 3c_2^2 + \dots + (n+1)c_n^2 = \frac{(n+2)\,.\,(2n-1)!}{n!\,(n-1)!}.$$

19. Find the sum of the coefficients in the expansion of

(i) $(1+2x)^5$; (ii) $(3x+7y)^6$; (iii) $(1+x+x^2)^4$.

20. Prove that

$$a + c_1(a+d)x + c_2(a+2d)x^2 + \dots + c_n(a+nd)x^n$$

equals $$(1+x)^{n-1}[a(1+x) + ndx].$$

21. Expand $x^p(1+x^q)^n$, where p, q, n are positive integers, in powers of x. What identity is obtained by differentiating w.r.t. x ? Deduce special results by putting (i) $x = 1$, (ii) $x = -1$.

22. Use the binomial theorem to prove that, if n is a positive integer, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$.

23. Obtain a relation from the identity

$$\{(1+x)^2-1\}^{2n} \equiv x^{2n}(2+x)^{2n}$$

by equating coefficients of x^{2n} in the expansions of the two sides.

24. Prove that $c_0+2^2c_1x+3^2c_2x^2+\dots+(n+1)^2c_nx^n$ equals

$$(1+x)^n+3nx(1+x)^{n-1}+n(n-1)x^2(1+x)^{n-2}.$$

25. Prove that $c_1+2^3.c_2x+3^3.c_3x^2+\dots+n^3c_nx^{n-1}$ equals

$$n(1+x)^{n-3}[1+(3n-1)x+n^2x^2].$$

26. If $(1+x+x^2)^n \equiv a_0+a_1x+a_2x^2+\dots+a_rx^r+\dots+a_{2n}x^{2n}$, obtain expansions by writing (i) $\frac{1}{x}$ for x, (ii) $-x$ for x, and show that $a_r=a_{2n-r}$.

What is the expansion of $(1+x^2+x^4)^n$?

Use the identity $(1+x^2+x^4)^n=(1+x+x^2)^n(1-x+x^2)^n$ to prove that $a_0^2-a_1^2+a_2^2-\dots+a_{2n}^2=a_n$.

27. With the data of No. 26, find the value of

(i) $a_0+a_2+a_4+\dots+a_{2n}$;

(ii) $a_0+2a_1+3a_2+\dots+(2n+1)a_{2n}$.

CHAPTER III
FINITE SERIES

Functional Notation

ANY expression, involving the variable x, whose value can be determined when the value of x is known, is called a **function of x**, and may be represented by the symbol $f(x)$ or $F(x)$ or $\varphi(x)$, etc. Thus $f(x)$ is shorthand for the words "function of x."

Suppose $f(x) \equiv 3x^4 - 4x^3 + 1$, then $f(2)$ represents the value of the function when $x = 2$.

Thus $\quad f(2) = 3(2)^4 - 4(2)^3 + 1 = 48 - 32 + 1 = 17.$

Similarly,

$$f(-1) = 3(-1)^4 - 4(-1)^3 + 1 = 3 + 4 + 1 = 8,$$
$$f(0) = 3(0)^4 - 4(0)^3 + 1 = 1,$$
$$f(x+h) = 3(x+h)^4 - 4(x+h)^3 + 1, \text{ and so on.}$$

Example 1. If $f(x) \equiv \dfrac{10x}{x^2 - 1} - 2x^3$, find the values of

$$\text{(i) } f(2x), \qquad \text{(ii) } f(x+1).$$

$$\text{(i) } f(2x) = \frac{10(2x)}{(2x)^2 - 1} - 2(2x)^3$$
$$= \frac{20x}{4x^2 - 1} - 16x^3.$$
$$\text{(ii) } f(x+1) = \frac{10(x+1)}{(x+1)^2 - 1} - 2(x+1)^3$$
$$= \frac{10(x+1)}{x^2 + 2x} - 2(x+1)^3.$$

Σ Notation

A series of terms is completely defined if the rth term is given as a function of r, for all positive integral values of r. This term may be represented by $f(r)$, but it is often more convenient to denote it by u_r. Also if

$$s_n = u_1 + u_2 + \ldots + u_r + \ldots + u_n,$$

s_n is called the *sum to n terms* of the series, and we write

$$s_n = \sum_{r=1}^{n} u_r \quad \text{or} \quad s_n = \sum_{1}^{n} u_r.$$

The Σ notation is a shorthand method of representing the sum of a number of terms of the same type.

Thus $$\sum_1^n r^2 \equiv 1^2+2^2+3^2+\ldots+n^2;$$

$$\sum_k^n r^2 \equiv k^2+(k+1)^2+(k+2)^2+\ldots+n^2;$$

$$\sum_1^n \frac{a_r}{x-a_r} \equiv \frac{a_1}{x-a_1}+\frac{a_2}{x-a_2}+\ldots+\frac{a_n}{x-a_n}.$$

EXERCISE III. a

1. If $f(x)\equiv x^3+3$, find the values of $f(1)$, $f(0)$, $f(-1)$, $f(2a)$, $f(3x)$.

2. If $\mathsf{F}(x)\equiv x^2-3x+2$, find the values of $\mathsf{F}(0)$, $\mathsf{F}(1)$, $\mathsf{F}(2)$, $\mathsf{F}\left(\frac{1}{x}\right)$, $\mathsf{F}(x-1)$.

3. If $\phi(x)\equiv 10^x$, find the values of $\phi(1)$, $\phi(2)$, $\phi(0)$, $\phi(-1)$, $\phi(-x)$.

4. Find the values of $\frac{f(x+h)-f(x)}{h}$ if (i) $f(x)\equiv x^2$, (ii) $f(x)\equiv\frac{1}{x}$.

What expressions are represented by the symbols in Nos. 5-10 ?

5. $\sum_1^n r(r+1)$. 6. $\sum_n^{2n} r!$. 7. $\sum_0^{n-1}(r+1)x^r$.

8. $\sum_1^{2n}(-1)^{r+1}r^2$. 9. $\sum_1^n\{(r+1)^3-r^3\}$.

10. $\sum_1^n f(r)$ where (i) $f(r)\equiv\frac{1}{r}$, (ii) $f(r)\equiv\log r$.

Use the Σ notation to express the sums of Nos. 11-14.

11. $1.3+2.4+3.5+4.6+\ldots$, n terms.

12. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, $2n$ terms.

13. $\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}$.

14. $f(a)-f(a+h)+f(a+2h)-\ldots$ n terms.

What are the values of the series in Nos. 15-21 ?

15. $\sum_1^n r$. 16. $\sum_n^{2n}(2r-1)$. 17. $\sum_{p=1}^n ar^{p-1}$.

18. $\sum_{r=1}^n \{a+(r-1)d\}$. 19. $\sum_m^n \log\frac{r+1}{r}$.

20. $\sum_1^n\{f(r+1)-f(r)\}$. 21. $\sum_m^n\{f(r-1)-f(r)\}$.

22. If $f(r)\equiv\frac{1}{3}r(r+1)(r+2)$, prove that

$$f(r)-f(r-1)=r(r+1).$$

Does this help you to find the value of $\sum_1^n r(r+1)$?

23. If $f(r)\equiv\frac{1}{r}$, prove that $f(r)-f(r+1)=\frac{1}{r(r+1)}$.

Does this help you to find the value of $\sum_1^m \frac{1}{r(r+1)}$?

24. Given that $\sum_1^n r^3=\frac{1}{4}n^2(n+1)^2$, find the value of $\sum_n^{2n} r^3$.

25. Given that $\sum_1^n r(r+1)=\frac{1}{3}n(n+1)(n+2)$, find the values of

$$\text{(i) } \sum_1^n r(r-1), \quad \text{(ii) } \sum_1^{n+1} r(r+1).$$

26. If u_r is the rth term of an A.P., prove that

$$u_n^2-2u_{n-1}^2+u_{n-2}^2$$

is independent of n.

Summation of Series

There is no general method for summing series, nor is it even always possible to express the sum to n terms as an elementary function of n. No such expression, for example, exists for the sum of the harmonical progression, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}$.

Formulae for the sums of Arithmetical and Geometrical Progressions have been given in Part III, *A New Algebra*, Ch. V, and various series involving binomial coefficients have been summed in Ch. II; we shall now discuss some other simple types.

The Difference Method

If it is possible to discover a function $f(r)$ such that

$$u_r\equiv f(r+1)-f(r),$$

it is easy to sum the series to n terms.

We have

$$u_1=f(2)-f(1);\ u_2=f(3)-f(2);\ u_3=f(4)-f(3);\ \ldots\ldots;$$
$$u_{n-1}=f(n)-f(n-1);\ u_n=f(n+1)-f(n).$$
$$\therefore\ u_1+u_2+u_3+\ldots+u_n=f(n+1)-f(1)$$

or
$$\sum_1^n u_r=f(n+1)-f(1).$$

Example 2. Sum to n terms

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots.$$

The rth term, $u_r=\dfrac{1}{r(r+1)}=\dfrac{1}{r}-\dfrac{1}{r+1}$.

$$\therefore\ u_1+u_2+u_3+\ldots+u_n$$
$$=(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

To sum the series

$$\mathbf{1^2+2^2+3^2+4^2+\ldots+n^2.}$$

The series Σr^2 can be summed by means of the identity

$$(r+\tfrac{1}{2})^3-(r-\tfrac{1}{2})^3=\text{etc.}$$

But to avoid fractions, we write

$$(2r+1)^3-(2r-1)^3=2\{3(2r)^2+1\}=24r^2+2.$$

If $r=1$, $24\,.\,1^2+2=3^3-1^3$.
If $r=2$, $24\,.\,2^2+2=5^3-3^3$.
If $r=3$, $24\,.\,3^2+2=7^3-5^3$.
....
If $r=n$, $24\,.\,n^2+2=(2n+1)^3-(2n-1)^3$.

$\therefore$ by addition,

$$24(1^2+2^2+\ldots+n^2)+2n=(2n+1)^3-1;$$
$$\therefore\ 24\sum_1^n r^2=8n^3+12n^2+4n;$$
$$\therefore\ \sum_1^n r^2=\tfrac{1}{6}n(2n^2+3n+1);$$
$$\therefore\ \sum_1^n \mathbf{r^2}=\tfrac{1}{6}\mathbf{n(n+1)(2n+1)}.$$

For an *alternative* method, see p. 40.

To sum the series

$$\mathbf{1^3+2^3+3^3+\ldots+n^3.}$$

$$\{r(r+1)\}^2-\{(r-1)r\}^2=(r^2+r)^2-(r^2-r)^2=4r^3.$$

Thus $4r^3=f(r+1)-f(r)$ where $f(r)\equiv\{(r-1)r\}^2$.

$\therefore$ as before, putting $r=1, 2, 3, \ldots, n$, and adding, we have

$$4\sum_1^n r^3=f(n+1)-f(1)=\{n(n+1)\}^2-0.$$

$$\therefore\ \sum_1^n \mathbf{r^3}=\{\tfrac{1}{2}\mathbf{n}(\mathbf{n}+1)\}^2.$$

It is easy to remember the sum of this series, because it is the square of the sum $(1+2+3+\ldots+n)$. Also the form of the sum suggests the "difference" which it is most convenient to use.

For alternative methods, see pp. 40, 42 and Ex. III. e, No. 26.

To sum the series

(i) $\mathbf{1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)}$;

(ii) $\mathbf{1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\ldots+n(n+1)(n+2)}$.

(i) $r(r+1)(r+2)-(r-1)r(r+1)$

$$=r(r+1)\{(r+2)-(r-1)\}=3r(r+1).$$

$\therefore$ as before, putting $r=1, 2, 3, \ldots, n$, and adding, we have

$$3\{1\cdot 2+2\cdot 3+\ldots+n(n+1)\}=n(n+1)(n+2)-0.$$

$$\therefore\ \sum_1^n \mathbf{r(r+1)}=\tfrac{1}{3}\mathbf{n(n+1)(n+2)}.$$

(ii) $r(r+1)(r+2)(r+3)-(r-1)r(r+1)(r+2)$

$$=r(r+1)(r+2)\{(r+3)-(r-1)\}=4r(r+1)(r+2).$$

$\therefore$ putting $r=1, 2, 3, \ldots, n$, and adding, we have

$$4\{1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)\}$$
$$=n(n+1)(n+2)(n+3)-0.$$

$$\therefore\ \sum_1^n \mathbf{r(r+1)(r+2)}=\tfrac{1}{4}\mathbf{n(n+1)(n+2)(n+3)}.$$

This method can obviously be applied to other series of this type. Thus we have

$$\sum_1^n r(r+1)(r+2)(r+3)=\tfrac{1}{5}n(n+1)(n+2)(n+3)(n+4),$$

and so on.

The results are easily remembered: *but it is advisable to check them*, whenever they are used, by putting $n=1$.

The same method can be applied to the more general series of this type :

(i) $a(a+d)+(a+d)(a+2d)+(a+2d)(a+3d)+\dots,$

(ii) $a(a+d)(a+2d)+(a+d)(a+2d)(a+3d)+\dots,$

and so on.

It is important to notice that in all these series the common difference between the first factors of successive terms is the same as the common difference between the factors of each term.

The results obtained above may be used for summing the series, Σr^2, Σr^3, Σr^4, etc.

Thus $\quad r^2\equiv r(r+1)-r.$

$$\begin{aligned}\therefore\ \sum_1^n r^2&=\sum_1^n r(r+1)-\sum_1^n r\\&=\tfrac{1}{3}n(n+1)(n+2)-\tfrac{1}{2}n(n+1)\\&=\tfrac{1}{6}n(n+1)(2n+1).\end{aligned}$$

Similarly, since $r(r+1)(r+2)\equiv r^3+3r^2+2r,$

$$\begin{aligned}r^3&\equiv r(r+1)(r+2)-3r^2-2r\\&\equiv r(r+1)(r+2)-3r(r+1)+r.\end{aligned}$$

$$\begin{aligned}\therefore\ \sum_1^n r^3&=\tfrac{1}{4}n(n+1)(n+2)(n+3)-3\times\tfrac{1}{3}n(n+1)(n+2)+\tfrac{1}{2}n(n+1)\\&=\tfrac{1}{4}n(n+1)\{(n^2+5n+6)-(4n+8)+2\}\\&=\tfrac{1}{4}n(n+1)(n^2+n)=\{\tfrac{1}{2}n(n+1)\}^2.\end{aligned}$$

Example 3. Sum to n terms

$$1\,.\,4\,.\,7+2\,.\,5\,.\,8+3\,.\,6\,.\,9+\dots.$$

$$\begin{aligned}u_r&=r(r+3)(r+6)=r(r^2+9r+18)\\&=r\{(r+1)(r+2)+6r+16\}\\&=r\{(r+1)(r+2)+6(r+1)+10\}\\&=r(r+1)(r+2)+6r(r+1)+10r.\end{aligned}$$

$$\begin{aligned}\therefore\ \sum_1^n u_r&=\tfrac{1}{4}n(n+1)(n+2)(n+3)\\&\qquad+6\times\tfrac{1}{3}n(n+1)(n+2)+10\times\tfrac{1}{2}n(n+1)\\&=\tfrac{1}{4}n(n+1)\{(n^2+5n+6)+(8n+16)+20\}\\&=\tfrac{1}{4}n(n+1)(n^2+13n+42)\\&=\tfrac{1}{4}n(n+1)(n+6)(n+7).\end{aligned}$$

EXERCISE III. b

Sum the following series :

1. $1^2+2^2+3^2+\dots+40^2$.
2. $1^3+2^3+3^3+\dots+20^3$.
3. $1^2+3^2+5^2+\dots+99^2$.
4. $21^3+22^3+\dots+50^3$.
5. $2^2+4^2+6^2+\dots+(4n)^2$.
6. $1^2+3^2+5^2+\dots+(4n-1)^2$.
7. $1^3+3^3+5^3+\dots+(2n-1)^3$.
8. $(n+1)^3+(n+2)^3+\dots+(2n)^3$.
9. Express in the form $ar(r+1)+br+c$,
   (i) r^2-3r; (ii) $3r^2+r+1$; (iii) $(r+3)(r-1)$.
10. Express in the form $ar(r+1)(r+2)+br(r+1)+cr+d$,
   (i) r^3+r; (ii) $(r+1)(r+3)(r+5)$.

Find the values of the following sums :

11. $\sum_1^n r(r+3)$.
12. $\sum_1^n r(r^2-1)$.
13. $\sum_1^n (r+1)^3$.
14. $\sum_1^n r(r+1)(2r+1)$.
15. $\sum_1^n r(r+1)(r+3)(r+4)$.

Sum to n terms the following series :

16. $1.3+2.4+3.5+\dots$.
17. $1.3+3.5+5.7+\dots$.
18. $1.11+2.12+3.13+\dots$.
19. $1^2.3+2^2.4+3^2.5+\dots$.
20. $1.2^2+2.3^2+3.4^2+\dots$.
21. $1.2.4+2.3.5+3.4.6+\dots$.
22. $1^2-2^2+3^2-4^2+\dots$ if n is (i) even, (ii) odd.
23. Evaluate $\sum_{r=1}^{n}(n-r)(r-1)$.
24. Evaluate $\sum_{r=1}^{n} r(n-r+1)^2$.

25. Equal spheres are piled in a triangular pyramid so that the numbers in successive layers or "*courses*," from the top downwards, are 1, 1+2, 1+2+3, 1+2+3+4, ... Find the number of spheres in a pile of n courses.

26. Equal spheres are piled in a pyramid with a square base. Find the number of spheres in an incomplete pyramid which has n courses, if each side of the base contains $2n$ spheres.

27. Find the number of spheres in a pile on a rectangular base whose sides contain respectively 15 and 20 spheres, if the top consists of a single row, all the spheres being equal.

28. Prove that the sum of the products in pairs of the first n integers is $\frac{1}{24}n(n^2-1)(3n+2)$.

29. Prove that the sum of the products in pairs of the first n odd integers is $\frac{1}{6}n(n-1)(3n^2-n-1)$.

30. If $s_1 = \sum_{r=1}^{n} (a + rd)$ and if $s_2 = \sum_{r=1}^{n} (a + rd)^2$, prove that

$$12(ns_2 - s_1^2) = n^2(n^2 - 1)d^2.$$

31. Use the identity $r^3(r+1)^3 - (r-1)^3r^3 \equiv 6r^5 + 2r^3$ to evaluate $\sum_1^n r^5$.

32. If $f(r) \equiv r^2(r-1)^2(2r-1)$, simplify $f(r+1) - f(r)$. Hence evaluate $\sum_1^n r^4$.

Method of Induction

The principle of this method has been explained on p. 22. It can only be employed for proving a *stated* result.

Example 4. Prove by induction that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

We first show that if the formula is true for any special value $n = k$, then it is also true for $n = k + 1$.

Suppose that $\qquad \sum_1^k r^3 = \tfrac{1}{4}k^2(k+1)^2.$

Then

$$\begin{aligned} \sum_1^{k+1} r^3 &= \sum_1^k r^3 + (k+1)^3 \\ &= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \tfrac{1}{4}(k+1)^2\{k^2 + 4(k+1)\} \\ &= \tfrac{1}{4}(k+1)^2(k+2)^2. \end{aligned}$$

$\therefore$ if the formula is true for $n = k$, it is also true for $n = k + 1$. But it is true for $n = 1$, because $1^3 = \tfrac{1}{4} . 1^2 . 2^2$; $\therefore$ it is true for $n = 2$, $\therefore$ for $n = 3$, and so on. Therefore the formula is true for all positive integral values of n.

The application of induction to summation of series is substantially equivalent to the difference method. For *if* the statement

$$u_1 + u_2 + u_3 + \dots + u_n = f(n)$$

is true for all positive integral values of n, then

$$u_1 + u_2 + u_3 + \dots + u_{n-1} = f(n-1).$$

$\therefore$ by subtraction, $\quad u_n = f(n) - f(n-1).$

This gives the necessary "difference" relation, and when it has been proved to be true, the work can be set out in the ordinary difference-manner.

EXERCISE III. c

Prove the following results by induction :

1. $\sum\limits_1^n r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

2. $\sum\limits_1^n r(r+2) = \frac{1}{6}n(n+1)(2n+7)$.

3. $\sum\limits_1^n \frac{1}{r(r+1)} = \frac{n}{n+1}$.

4. $\sum\limits_1^n (2r-1)^2 = \frac{1}{3}n(4n^2-1)$.

5. $\sum\limits_1^n \frac{1}{4r^2-1} = \frac{n}{2n+1}$.

6. $\sum\limits_1^n \frac{1}{(3r-2)(3r+1)} = \frac{n}{3n+1}$.

7. $1^2 - 2^2 + 3^2 - 4^2 + \ldots$ to n terms $= \frac{1}{2}(-1)^{n-1}n(n+1)$.

8. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots$ to n terms $= (n+1)! - 1$.

9. $\sum\limits_1^n (r^3 - r) = \frac{1}{4}n(n+1)(n+2)(n-1)$.

10. $\sum\limits_1^n \frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(n+1)(n+2)}$.

11. $\sum\limits_1^n (r^2+1) \cdot r! = n \cdot (n+1)!$.

12. $\frac{8}{3 \cdot 5} - \frac{12}{5 \cdot 7} + \frac{16}{7 \cdot 9} - \ldots$ to n terms $= \frac{1}{3} + (-1)^{n-1} \cdot \frac{1}{2n+3}$.

13. If n and r are positive integers, prove that

$$1 + n + \frac{n(n+1)}{2!} + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}$$

equals
$$\frac{(n+1)(n+2)\ldots(n+r)}{r!}.$$

14. If $u_{r+1} = 2u_r + 1$ for all positive integral values of r, prove that $u_n + 1 = 2^{n-1}(u_1 + 1)$. Also find the value of $\sum\limits_1^n u_r$ if $u_1 = 1$.

15. If $4 = \frac{3}{u_1} = u_1 + \frac{3}{u_2} = u_2 + \frac{3}{u_3} = u_3 + \frac{3}{u_4} = \ldots$, prove that

$$u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}.$$

16. If $f(n) \equiv 3^{2n} + 7$ where n is a positive integer, prove that $f(n+1) - f(n)$ is divisible by 8. Hence prove that $3^{2n} + 7$ is divisible by 8.

17. If $f(n) \equiv (3n+1) . 7^n - 1$ where n is a positive integer, prove that $f(n+1) - f(n)$ is divisible by 9. Hence prove that $f(n)$ is divisible by 9.

18. If $u_1 = 0$ and if $u_{n+1} = (1-x)u_n + nx$ for all positive integral values of n, prove that $u_n = \frac{1}{x}\{nx - 1 + (1-x)^n\}$.

19. If $2u_1 = a + b$, $2u_2 = b + u_1$, $2u_3 = u_1 + u_2$, etc., prove that $3u_n = a\{1 - (-\frac{1}{2})^n\} + b\{2 + (-\frac{1}{2})^n\}$.

20. If s_r is the sum of k terms of an A.P. whose first term is r and common difference $2r - 1$, prove that $\sum_1^m s_r = \frac{1}{2}km(km+1)$.

Power Series

The simplest series in ascending powers of a variable x is the series in which all the coefficients are equal, namely the geometrical progression, $a + ax + ax^2 + \dots + ax^{n-1}$, whose sum is

$$\frac{a(1-x^n)}{1-x}.$$

Other power series have been discussed in connection with the binomial theorem in Ch. II. We shall now consider series where the coefficient of x^r is a rational integral function of r; the method used for summing a G.P. can be extended to such series. We first take the case when the coefficient of x^r is a linear function of r; this is sometimes called the arithmetico-geometric series. It will be found that the product of s_n and $1-x$ reduces to a G.P. together with extra terms at the beginning and end.

To sum the series

$$\mathbf{a + (a+d)x + (a+2d)x^2 + \dots + [a + (n-1)d]x^{n-1}}.$$

$$s_n = a + (a+d)x + (a+2d)x^2 + \dots + [a + (n-1)d]x^{n-1}.$$

$$\therefore\ xs_n = ax + (a+d)x^2 + \dots + [a + (n-2)d]x^{n-1} + [a + (n-1)d]x^n;$$

$$\therefore\ s_n - xs_n = a + dx + dx^2 + \dots + dx^{n-1} - [a + (n-1)d]x^n;$$

$$\therefore\ s_n(1-x) = a + \frac{dx(1-x^{n-1})}{1-x} - [a + (n-1)d]x^n;$$

$$\therefore\ s_n = \frac{a - [a + (n-1)d]x^n}{1-x} + \frac{dx(1-x^{n-1})}{(1-x)^2}.$$

Similarly, if the coefficient of x^r is a *quadratic* function of r, it will be found that the product of s_n and $(1-x)^2$ reduces to a G.P. together with extra terms at the beginning and end. If the coefficient of x^r is a *cubic* function of r, the same is true of the product of s_n and $(1-x)^3$; and so on.

Calculus methods may often be employed with advantage for series of this kind, as in the following example; see also p. 30.

Example 5. Sum the series,

$$s_n \equiv 1^2 + 2^2x + 3^2x^2 + \ldots + n^2x^{n-1}.$$

The coefficient of x^r in $(1-2x+x^2)s_n$ for values of r from 2 to $n-1$ is

$$r^2 - 2(r-1)^2 + (r-2)^2 = r^2 - 2r^2 + 4r - 2 + r^2 - 4r + 4 = 2;$$

$$\therefore\ s_n(1-x)^2 = 1 + x(4-2) + 2x^2 + 2x^3 + \ldots$$
$$+ 2x^{n-1} + x^n[(n-1)^2 - 2n^2] + n^2x^{n+1}.$$

$$\therefore\ s_n(1-x)^2 = 1 + \frac{2x(1-x^{n-1})}{1-x} + x^n(1-2n-n^2) + n^2x^{n+1};$$

$$\therefore\ s_n = \frac{2x(1-x^{n-1})}{(1-x)^3} + \frac{1-x^n(n^2+2n-1)+n^2x^{n+1}}{(1-x)^2}.$$

Alternatively, as follows:

$$x + x^2 + x^3 + \ldots + x^n = \frac{x(1-x^n)}{1-x} = \frac{x-x^{n+1}}{1-x}.$$

Differentiate w.r.t. x,

$$\therefore\ 1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1-(n+1)x^n}{1-x} + \frac{x-x^{n+1}}{(1-x)^2};$$

$$\therefore\ x + 2x^2 + 3x^3 + \ldots + nx^n = \frac{x-(n+1)x^{n+1}}{1-x} + \frac{x^2-x^{n+2}}{(1-x)^2}.$$

If we now differentiate again w.r.t. x, we shall obtain the given series and its sum.

EXERCISE III. d

Sum to n terms the series in Nos. 1-8:

1. $1 + 2x + 3x^2 + 4x^3 + \ldots$.
2. $1 - 3x + 5x^2 - 7x^3 + \ldots$.
3. $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \ldots$.
4. $1 - \frac{2}{3} + \frac{3}{3^2} - \frac{4}{3^3} + \ldots$.
5. $a + \frac{1}{2}(a+d) + \frac{1}{4}(a+2d) + \frac{1}{8}(a+3d) + \ldots$.
6. $x^{n-1} - 2x^{n-2}y + 3x^{n-3}y^2 - 4x^{n-4}y^3 + \ldots$.
7. $1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{1}{2}r(r+1)x^{r-1} + \ldots$.

8. $1+3x+7x^2+15x^3+\ldots+(2^r-1)x^{r-1}+\ldots$.

9. What is the coefficient of x^r in

$$(1-2x+x^2)\,.\sum_0^n(a+rd)x^r$$

for values of r from 2 to n inclusive ?

10. What is the coefficient of x^r in

$$(1-x)^3\,.\sum_0^n(a+br+cr^2)x^r$$

for values of r from 3 to n inclusive ?

11. What is the coefficient of x^r in $(1-x)^4\,.\sum_0^n r^3x^r$ for values of r from 4 to n inclusive ?

12. Find the value of $\sum_1^n(2^r-r^2)x^{r-1}$.

MISCELLANEOUS EXAMPLES

EXERCISE III. e

1. Find the least value of n such that the sum of the first n positive integers exceeds 1000.

2. Sum the series

$$1+\left(x^2+\frac{1}{x^2}\right)+\left(x^4+\frac{1}{x^4}\right)+\ldots+\left(x^{2n}+\frac{1}{x^{2n}}\right).$$

3. If $f(x)\equiv x+\frac{1}{x}$, prove that $\{f(x)\}^3-f(x^3)=3f(x)$.

4. If $u_r=2^{2r}-2r-1$, find $\sum_1^n u_r$.

Sum to n terms the series in Nos. 5-8:

5. $2\,.\,5+5\,.\,8+8\,.\,11+\ldots$. 6. $1\,.\,5+3\,.\,7+5\,.\,9+\ldots$.

7. $(n+1)^2+(n+2)^2+(n+3)^2+\ldots$.

8. $(2n)^3+(2n-1)^3+(2n-2)^3+\ldots$.

9. If n is odd, prove that $1+x^2+x^4+\ldots+x^{2n-2}$ equals

$$(1+x+x^2+\ldots+x^{n-1})(1-x+x^2-x^3+\ldots+x^{n-1}).$$

10. Find the sum of the series,

$$1\,.\,n+2(n-1)+3(n-2)+4(n-3)+\ldots+n\,.\,1.$$

Prove the statements in Nos. 11-16.

11. $\sum_1^n\frac{r}{(r+1)!}=1-\frac{1}{(n+1)!}$. 12. $\sum_1^n\frac{r-(r-1)^2}{r!}=1+\frac{n-1}{n!}$.

13. $\sum_1^n \frac{r \cdot 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$. 14. $\sum_1^n \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$

15. $\sum_1^n r^2(r+1) = \frac{1}{12}n(n+1)(n+2)(3n+1)$.

16. $\sum_1^n \frac{2r}{1+r^2+r^4} = 1 - \frac{1}{1+n+n^2}$.

17. If $u_n \equiv 2^{n-1} - 3n + 5$, prove that

$$u_n = 4u_{n-1} - 5u_{n-2} + 2u_{n-3}.$$

18. If $u_1, u_2, u_3, \ldots$ form a G.P. with common ratio k, find in terms of k and u_1 the value of $\sum_1^n (u_r u_{r+1})$.

19. Sum to n terms the series

$$1.3.5 + 2.4.6 + 3.5.7 + \ldots.$$

20. Prove that $\sum_1^n r \log \frac{r+1}{r} = \log \frac{(n+1)^n}{n!}$.

21. Prove that the sum to n terms of the series

$$1 + x(1+x) + x^2(1+x+x^2) + x^3(1+x+x^2+x^3) + \ldots$$

equals $$\frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}.$$

22. Prove that $\sum_1^n (r^3 + 3r^5) = \frac{1}{2}n^3(n+1)^3$.

23. The population of a town increases in such a way that if it is N at the beginning of a year, then at the end of the year it is $a + b\text{N}$, where a, b are constants. Show that, after the lapse of k years from the time when it was N, it is $\frac{a}{1-b} + \left(\text{N} - \frac{a}{1-b}\right)b^k$.

24. If $f(x) \equiv (x^2 - x + 1)^3/(x - x^2)^2$, find the values of

$$f(1-x) \text{ and } f\left(\frac{1}{x}\right)$$

and deduce those of

$$f\left(1 - \frac{1}{x}\right),\ f\left(\frac{1}{1-x}\right) \text{ and } f\left(\frac{x}{x-1}\right).$$

25. Prove that $\frac{2(2x-1)}{x(x+1)(x+2)} = -\frac{1}{x} + \frac{6}{x+1} - \frac{5}{x+2}$.

Hence sum to n terms the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \ldots.$$

26. Find the number of terms in the first $(n-1)$ brackets of the series,

$$(1)+(3+5)+(7+9+11)+(13+15+17+19)+\dots.$$

What is the first term of the nth bracket? What is the sum of the series in the nth bracket?

Hence evaluate $\sum_1^n r^3$.

27. If $\dfrac{u_n u_{n+1}+3a(u_n+u_{n+1})-3a^2}{u_n u_{n+1}-3a(u_n+u_{n+1})+3a^2}=\frac{1}{2}n$, for all positive integral values of n, prove that

$$\frac{u_n u_{n+1}}{(u_n-a)(u_{n+1}-a)}=\frac{3(n+2)}{2(n+4)}.$$

Hence prove that

$$\frac{u_1 u_{2n}}{(u_1-a)(u_{2n}-a)}=\frac{9(n+1)}{4(2n+3)},$$

and that

$$\frac{u_1 u_{2n}+3a(u_1+u_{2n})-3a^2}{u_1 u_{2n}-3a(u_1+u_{2n})+3a^2}=\frac{2n}{n+3}.$$

28. Simplify $\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}$. Hence evaluate

$$\sum_1^n \frac{1}{r(r+1)(r+2)}.$$

29. Simplify

$$\frac{1}{r(r+1)(r+2)}-\frac{1}{(r+1)(r+2)(r+3)}.$$

Hence evaluate $$\sum_1^n \frac{1}{r(r+1)(r+2)(r+3)}.$$

30. If $f(r)\equiv 3r^4(r-1)^4-4r^3(r-1)^3$, simplify $f(r+1)-f(r)$ and deduce the value of $\sum_1^n r^7$.

31. If $f(r)\equiv r^3(r-1)^3(2r-1)-r^2(r-1)^2(2r-1)$, simplify $f(r+1)-f(r)$ and deduce the value of $\sum_1^n r^6$.

TEST PAPERS A. 1–10

A. 1

1. If $V=\dfrac{1}{3}\sqrt{\left(\dfrac{S^3}{8\pi}\right)}$,

(i) find V when $S=73{\cdot}45$; (ii) find S when $V=107{\cdot}2$.

2. If α, β are the roots of $2x^2-3x=7$, find the equation whose roots are $\dfrac{\alpha}{\beta}$, $\dfrac{\beta}{\alpha}$.

3. Find (i) the coefficient of x^n in $(1-x+x^2)(1+x)^{2n+1}$;

(ii) the term independent of x in $\left(x-\dfrac{1}{x^2}\right)^{12}$.

4. In how many ways can 3 cards be selected from a pack of 52 cards? In how many of these ways will the cards be of different suits.

5. (i) If $u_r = r(2r+1)+2^{r+1}$, find the value of $\sum\limits_1^n u_r$.

(ii) Express in factorials ${}_{4n}C_n \cdot {}_{3n}C_{2n}$.

A. 2

1. (i) If $n = \log 3 \div \log\left(1+\dfrac{r}{100}\right)$, find n when $r = 3\frac{1}{2}$.

(ii) Simplify $(x^{\frac{1}{2}}-1+x^{-\frac{1}{2}})(x^{\frac{1}{2}}+1+x^{-\frac{1}{2}})$.

2. Find c if $x+1$ is a factor of x^3+cx^2-5x-6, and in this case find the other factors.

3. (i) If $\sum\limits_1^n u_r = 2n^2+3n$ for all values of n, find u_n and the value of $\sum\limits_{n+1}^{2n} u_r$.

(ii) Sum to $2n$ terms, $2.5+3.6+4.7+\ldots$.

4. How many numbers greater than 6000 can be formed with the digits 3, 4, 5, 6, 8 if no digit is repeated? How many of these are odd numbers?

5. (i) Find the coefficients of x^2 and x^3 in $(x-1)^{2n}(x+1)^n$.

(ii) Simplify $\{(n+1)!-(n-1)!\} \div n!$.

A. 3

1. (i) If $r = \frac{2}{3}$, find the least integral value of n for which $\dfrac{r^n}{1-r}$ is less than 0·0001.

(ii) Evaluate x^2+4x when $x = \sqrt{5}-2$.

2. The sum of 50 terms of an A.P. is 200, and the sum of the next 50 terms is 2700, find the common difference and the first term.

3. (i) Simplify

$$(x+2y)^5-5y(x+2y)^4+10y^2(x+2y)^3 \\ -10y^3(x+2y)^2+5y^4(x+2y).$$

(ii) Evaluate $(1{\cdot}02)^{10}$ to 4 places of decimals.

4. In how many ways can 4 boys and 3 girls be arranged in a line if the girls are not separated.

5. (i) Sum to n terms, $3+4x+5x^2+6x^3+\ldots$.

(ii) Sum to $2n$ terms, $1^2+2^3+3^2+4^3+5^2+6^3+\ldots$.

A. 4

1. (i) If $ax^2=b+\log\sqrt{c}$, find x when $a=3{\cdot}142$, $b=1{\cdot}895$, $c=128{\cdot}6$.

(ii) Simplify $\log 8\div\log\frac{1}{8}$.

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{a^4-e^4}{b^4-f^4}=\frac{(a^2-c^2)(c^2-e^2)}{(b^2-d^2)(d^2-f^2)}$.

3. In how many ways can the letters in *tobacco* be arranged? In how many of these ways are there exactly 3 letters between the c's?

4. Which is the greatest coefficient in the expansion of $(3x+4y)^9$? What is the difference between the sums of the odd and the even coefficients?

5. Sum to n terms,

(i) $n(n+1)+(n+1)(n+2)+(n+2)(n+3)+\ldots$.

(ii) $\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\ldots$.

A. 5

1. (i) If $W\propto r^3$ and if $S\propto r^2$, what is the effect on W of doubling S?

(ii) Find $x:y:z$ if $2x-3y=3z$ and $2x+y=5z$.

2. If α, β are the roots of $x^2+qx+r=0$, find the condition that (i) $2\alpha=3\beta$; (ii) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{1}{\alpha\beta}$.

3. If s_n denotes the sum of the first n terms of an A.P., prove that its common difference is $s_n-2s_{n+1}+s_{n+2}$.

4. In how many ways can 12 people take their seats in a row of 12 chairs if two of them must be separated and if two others must sit next to each other?

5. (i) Find the coefficient of $x^{2n}y^nz^n$ in $(x+y+z)^{4n}$.

(ii) Find the coefficient of x^{2n-2} in $(1-x+2x^2)^n$.

(iii) Prove that ${}_{2n+1}C_1+{}_{2n+1}C_2+{}_{2n+1}C_3+\ldots$, n terms, $=4^n-1$.

A. 6

1. Solve (i) $8^x=16$; (ii) $8^x=13$.

2. Find the values of a, b, c if, for all values of x,

$$\frac{20x}{(x+1)(x-2)(x+3)}=\frac{a}{x+1}+\frac{b}{x-2}+\frac{c}{x+3}.$$

3. How many even numbers between 100 and 1000 can be formed with the digits 4, 5, 6, 7, 8, if no digit is repeated? How many, if digits may be repeated?

4. If $u_1=0$ and if $u_{r+1}=(1+x)u_r-rx$ for all positive integral values of r, prove by induction, or otherwise, that

$$u_n=\frac{1}{x}[1+nx-(1+x)^n].$$

5. (i) What is the term independent of x in the expansion of $(1+x)^p\left(1+\frac{1}{x}\right)^q$?

(ii) If $(1+x)^n\equiv c_0+\Sigma c_r x^r$, prove that

$$c_0c_2+c_1c_3+c_2c_4+\ldots+c_{n-2}c_n=\frac{(2n)!}{(n+2)!\,(n-2)!}.$$

A. 7

1. Write in a form not involving logarithmic notation,

(i) $2\log a-3\log b=2$; (ii) $c\log 3=4$; (iii) $\log x=y\log 2+1$.

2. What is the greatest value of

(i) $5-(2x-1)^2$; (ii) $8x-3x^2$?

3. How many integers are factors of $2^4.3^2.5^3.7$, excluding 1 and the number itself?

4. (i) If $f(r)=\frac{1}{r^2}$, prove that $f(r)-f(r+1)=\frac{2r+1}{r^2(r+1)^2}$, and deduce the sum to n terms of

$$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots.$$

(ii) Evaluate $\sum_1^n(n-r)(r+2)$.

5. (i) Find the greatest term in the expansion of $(3+5x)^{10}$ when $x=\frac{1}{2}$.

(ii) If $(1+x)^n\equiv 1+\Sigma c_r x^r$, find the value of

$$2c_1-3c_2+4c_3\ldots+(-1)^{n+1}(n+1)c_n.$$

A. 8

1. Solve, correct to two places of decimals,

(i) $x+x\sqrt{3}=8$; (ii) $x^3+x\sqrt{3}=8$.

2. If $(s-a):(s-b):(s-c)=1:7:4$ and if $s=\frac{1}{2}(a+b+c)$, find $a:b:c$.

3. (i) Find the sum to n terms of

$$1+(3+5+7)+(9+11+13+15+17)+\ldots.$$

(ii) Find a relation between p, q, r if these numbers are the 2nd, 9th, 11th terms of a G.P.

4. In how many ways can n unlike things be arranged in a row, $n > 6$, so that 4 particular things are separated ?

5. (i) Find the value of $\sum_1^n {}_nC_r x^r (1+x+x^2)^{n-r}$.

(ii) Prove that $3^{2n+4} - 2^{2n}$ is a multiple of 5, if n is a positive integer.

A. 9

1. Factorise

$$(b+c)\{(c+a)^2-(a+b)^2\}+(c+a)\{(a+b)^2-(b+c)^2\}$$
$$+(a+b)\{(b+c)^2-(c+a)^2\}.$$

2. (i) Solve $\sqrt{(2x+1)}+\sqrt{(2x-2)}=3$.

(ii) Simplify $9^{2n+1}.\, 3^{1-n} \div 27^{n+1}$.

3. There are 3 sets of parallel lines containing respectively p lines, q lines, r lines. What is the greatest number of parallelograms that can be formed by the system ?

4. (i) Prove that $\sum_1^n \frac{r+2}{r(r+1)}(\frac{1}{2})^r = 1 - \frac{1}{n+1}(\frac{1}{2})^n$.

(ii) Sum to n terms, (i) if n is even, (ii) if n is odd,

$$1^3 - 2^3 + 3^3 - 4^3 + \dots .$$

5. Prove that

$$1 - {}_nC_1 \frac{1+x}{1+nx} + {}_nC_2 \frac{1+2x}{(1+nx)^2} - {}_nC_3 \frac{1+3x}{(1+nx)^3} + \dots = 0.$$

A. 10

1. Simplify (i) $\sqrt{(73-40\sqrt{3})}$; (ii) $\log 125 \div \log 25$.

2. Prove that the value of $\frac{4x+5}{(x+1)(x-1)}$ cannot be between -1 and -4. Sketch the graph of this function.

3. (i) In how many ways can 12 people be arranged at 4 similar round tea-tables, each of which holds 3 people ?

(ii) There are p letters a, q letters b, r letters c. How many selections of k letters can be made from them if (i) $k < p < q < r$, (ii) $p < k < q < r$?

4. Prove that the sum to n terms of the series,

$$\frac{1}{2} \cdot \frac{1}{1^2 \cdot 3^2} + \frac{1}{3} \cdot \frac{1}{2^2 \cdot 4^2} + \frac{1}{4} \cdot \frac{1}{3^2 \cdot 5^2} + \dots ,$$

is

$$\frac{1}{16} - \frac{1}{4(n+1)^2(n+2)^2}.$$

5. If $(1+x)^n \equiv c_0 + \Sigma c_r x^r$,

(i) prove that $\frac{1}{2}c_1 + \frac{1}{4}c_3 + \frac{1}{6}c_5 + \dots = \frac{2^n - 1}{n+1}$;

(ii) find the value of $c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 + \dots + (-1)^n \frac{1}{n+1} c_n$.

CHAPTER IV

LIMITS AND CONVERGENCE

Sum of a Series

We have investigated in Ch. III expressions for the sum to n terms of various series, using the notation

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

Here n is necessarily a positive integer, and s_n is called a function of the *positive integral variable, n.* The object of this chapter is to discuss the behaviour of

$$s_n \equiv f(n)$$

where n is allowed to take *integral* values which increase indefinitely, that is, to take the values 1, 2, 3, 4, 5, ... and so on without end.

The special case, where u_1, u_2, u_3, ... are terms of a G.P. has been examined in detail in Part III, *A New Algebra*, Ch. VI, pp. 77-79, and illustrates the various ways in which s_n can behave. We shall now consider some further examples.

To avoid tedious repetition, the convention will be made that whenever the letter n is used in this chapter, it denotes a *positive integer.*

The Idea of a Limit

Consider $f(n) \equiv 3 + \dfrac{1}{n}$.

Suppose n takes in succession the values

2 5 10 50 100 1000 100000, etc.

then the values of $3 + \dfrac{1}{n}$ are respectively

3·5 3·2 3·1 3·02 3·01 3·001 3·00001, etc.

As n increases, $3 + \dfrac{1}{n}$ steadily decreases; and the larger n becomes, the more closely the value of $3 + \dfrac{1}{n}$ approaches 3, and

for all sufficiently large values of n, the value of $3+\frac{1}{n}$ is as near 3 as we please. This behaviour is described by the statement:

$3+\frac{1}{n}$ tends to 3 when n tends to infinity,

or

the limit of $3+\frac{1}{n}$ is 3 when n tends to infinity.

For brevity, this is written

$$3+\frac{1}{n}\to 3 \text{ when } n\to\infty \quad \text{or} \quad \lim_{n\to\infty}\left(3+\frac{1}{n}\right)=3.$$

Although we say that the "limit" of $3+\frac{1}{n}$ *is* 3 when $n\to\infty$, there is no value of n for which $3+\frac{1}{n}$ *equals* 3.

The reader should *never* say $\frac{1}{n}=0$ if $n=\infty$ or $\frac{1}{\infty}=0$, because the word "infinity," or its symbol ∞, as used above, does not represent a number; the phrase, "n tends to infinity" is merely a short way of saying that there is no restriction on the size of the values n is allowed and supposed to take.

Next consider $f(n)\equiv 3-\frac{1}{n}$.

This function steadily increases as n increases; and the larger n becomes, the more closely the value of $3-\frac{1}{n}$ approaches 3, and for all sufficiently large values of n, its value is as near 3 as w please. We therefore say

$$3-\frac{1}{n}\to 3 \text{ when } n\to\infty \quad \text{or} \quad \lim_{n\to\infty}\left(3-\frac{1}{n}\right)=3,$$

although there is no value of n for which $3-\frac{1}{n}$ equals 3.

A function can, of course, tend to a limit without steadily increasing or steadily decreasing. Thus

$$f(n)\equiv 3+(-1)^n\frac{1}{n}$$

is greater than 3 if n is even and is less than 3 if n is odd, but for all sufficiently large values of n, its value is as near 3 as we please.

$$\therefore\ 3+(-1)^n\frac{1}{n}\to 3 \text{ when } n\to\infty.$$

In general, if $f(n)$ is a function of the positive integral variable n, and if, for ALL sufficiently large values of n, the difference between $f(n)$ and a constant l is as small as we please, we say that

$$f(n) \to l \text{ when } n \to \infty \quad \text{or} \quad \lim_{n\to\infty} f(n) = l.$$

This means that the person who names the difference can choose as small a number as he likes, small *not* zero, but he must speak *first*. After he has spoken it rests with his opponent to find a number N such that the difference between $f(n)$ and l falls below the named amount *whenever* $n > N$. It does not mean that the second man must be able to find in advance a number N which will suit every conceivable small difference that can afterwards be chosen.

The word ALL is important, because it is not enough that *some* sufficiently large values should exist.

Non-existence of Limits

There are several ways in which a function $f(n)$ may behave when $n \to \infty$.

Thus the function $3n^2 - 1000$ takes values as large as we please for *all* sufficiently large values of n; it does not therefore tend to a "limit," in the sense in which this word is used above. And we write

$$3n^2 - 1000 \to \infty \quad \text{when} \quad n \to \infty.$$

Similarly $1000 - 3n^2$ takes negative values, *numerically* as large as we please, for *all* sufficiently large values of n, and so we write

$$1000 - 3n^2 \to -\infty \text{ when } n \to \infty.$$

Again, the function $1 + (-1)^{n+1}n^2$ is positive when n is odd and is negative when n is even, and takes values *numerically* as large as we please, both positive and negative, for *all* sufficiently large values of n. We therefore say that

$$1 + (-1)^{n+1}n^2 \textit{ oscillates infinitely} \text{ when } n \to \infty.$$

Similarly we say that

$$f(n) \equiv 3 + \frac{1}{n} + n\{1 + (-1)^n\}$$

oscillates infinitely, although it is never negative, because

$f(n) \to \infty$ when $n \to \infty$ through even values, and

$f(n) \to 3$ when $n \to \infty$ through odd values.

In all these cases, no limit of $f(n)$, when $n \to \infty$, exists. It does not even follow that a limit exists when all the values of $f(n)$ lie between fixed boundaries. Thus if

$$f(n) \equiv 3 + \frac{1}{n} + 2(-1)^n,$$

no value of $f(n)$ is less than 1 and no value is greater than $5\frac{1}{2}$. But $\lim\limits_{n \to \infty} f(n)$ does not exist, because when n is large and odd, $f(n)$ takes values near 1, and when n is large and even, $f(n)$ takes values near 5, and we say that the function *oscillates finitely* when $n \to \infty$.

Behaviour of x^n when $n \to \infty$

The symbolism introduced above may now be used to express concisely the important results established in Part III, *A New Algebra*, Ch. VI, pp. 78-79.

If $-1 < x < 1$, $x^n \to 0$ when $n \to \infty$,
if $x > 1$, $x^n \to \infty$ when $n \to \infty$,
if $x < -1$, x^n oscillates infinitely when $n \to \infty$,
if $x = -1$, x^n oscillates finitely when $n \to \infty$.

EXERCISE IV. a

1. For what values of n does $\frac{3n+1}{n+2}$ differ from 3 by less than 0·01?

2. For what values of n does $\frac{n^2+5}{2n^2+1}$ differ from $\frac{1}{2}$ by less than 0·001?

3. For what values of n does $\frac{10^n}{10^n-1}$ differ from 1 by less than 0·0001?

4. For what values of n is

$$\text{(i)}\ \frac{n}{n^2+1} < 0{\cdot}001\,; \quad \text{(ii)}\ \frac{n^2}{n-1} > 1000\,?$$

5. For what values of n is $n^2 - 10n$ greater than

(i) 144, (ii) 1000000?

6. Find the least integer k such that $\frac{n}{100} + (-1)^n$ exceeds 10 *whenever* $n > k$. What is the least value of n for which the expression exceeds 10?

7. Can you find an integer k such that $n+(-1)^n n$ exceeds 100 whenever $n>k$?

Discuss the behaviour, when $n \to \infty$, of the functions in Nos. 8-22:

8. $1+\frac{1}{n^2}$.
9. $2-\frac{1000}{n}$.
10. $\frac{3n+4}{n+1}$.
11. $\frac{1}{n}\{n+(-1)^n\}$.
12. $(-1)^n$.
13. $\frac{1}{n}\{2n(-1)^n+1\}$.
14. n^2-2n.
15. $n+(-1)^n$.
16. $n\{1+(-1)^n\}$.
17. $n\{2+(-1)^n\}$.
18. $\frac{2^n}{1+2^n}$.
19. $\frac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$.
20. $(-\frac{1}{2})^n$.
21. $(-1{\cdot}1)^n$.
22. $n+(-1)^n n^2$.

Discuss the behaviour when $n \to \infty$ of s_n where s_n denotes the sum to n terms of the series in Nos. 23-32:

23. $1+2+3+4+\dots$.
24. $1-1+1-1+\dots$.
25. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$.
26. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots$.
27. $(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\dots$.
28. $1-2+3-4+5-6+7-\dots$.
29. $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\dots$.
30. $(2-1)-(2-\frac{1}{2})+(2-\frac{1}{4})-(2-\frac{1}{8})+\dots$.
31. $1-2-3+4+5-6-7+8+9-10-\dots$.
32. $\left(1+\frac{1}{3}\right)-\left(1+\frac{1}{3^2}\right)+\left(\frac{1}{2}+\frac{1}{3^3}\right)-\left(\frac{1}{2}+\frac{1}{3^4}\right)+\left(\frac{1}{3}+\frac{1}{3^5}\right)-\left(\frac{1}{3}+\frac{1}{3^6}\right)+\dots$.

Infinite Series

If u_r is a given function of r, the series,

$$u_1,\ u_2,\ u_3,\ \dots,\ u_r,\ \dots,$$

in which there is a first term and every term has an immediate successor, is called a sequence or progression, and is the only kind of infinite series with which we shall be concerned in this book.

The sum of the first n terms of the series, namely

$$u_1+u_2+u_3+\dots+u_n,$$

is denoted by s_n, and the series itself by Σu_r.

If s_n tends to a limit s, when n tends to infinity, the series Σu_r is called **convergent** and s is called its **sum to infinity**. (Cf. *New Algebra*, Part III, p. 79.)

Thus, if $u_r \equiv \dfrac{1}{r(r+1)}$,

$$s_n = \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \ldots + \frac{1}{n(n+1)}$$
$$= (1-\tfrac{1}{2}) + (\tfrac{1}{2}-\tfrac{1}{3}) + (\tfrac{1}{3}-\tfrac{1}{4}) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}.$$

But $s_n = 1 - \dfrac{1}{n+1} \to 1$ when $n \to \infty$;

$\therefore$ this series is convergent and its "sum to infinity" is 1.

The *sum to infinity* is not a sum in the ordinary sense, because the word "sum" means the result obtained from a process of successive addition which eventually terminates. The operation of adding up the terms of an infinite series, one by one, can never be completed. However many times the operation is repeated, the process remains unfinished. But if a series is convergent, the successive sums tend to a limit, and the sum to infinity is *not* "a sum" but "the limit of a sum."

If $$s_n = u_1 + u_2 + u_3 + \ldots + u_n,$$
and if $s_n \to \infty$ when $n \to \infty$, the series Σu_r is called **divergent** and is said to diverge to ∞.

If $s_n \to -\infty$ when $n \to \infty$, Σu_r is also called divergent and is said to diverge to $-\infty$.

Thus, if $u_r = r$,

$$s_n = 1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1);$$
$$\therefore\ s_n \to \infty \text{ when } n \to \infty;$$
$$\therefore\ \Sigma r \text{ is divergent and diverges to } \infty.$$

Similarly $\Sigma(-r)$ is divergent and diverges to $-\infty$.

If s_n oscillates finitely or infinitely when $n \to \infty$, the series Σu_r is called an **oscillating series**, and is said to oscillate finitely or infinitely.

Thus, if $u_r = (-1)^{r+1}$,

$$s_n = 1 - 1 + 1 - 1 + \ldots\ n \text{ terms};$$
$$\therefore\ s_n = 1 \text{ if } n \text{ is odd and } s_n = 0 \text{ if } n \text{ is even};$$
$$\therefore\ \Sigma(-1)^{r+1} \text{ oscillates finitely.}$$

Also, if $u_r = (-1)^{r+1}r$,

$$s_n = 1 - 2 + 3 - 4 + 5 - \dots \; n \text{ terms};$$

$$\therefore\; s_n = -\tfrac{1}{2}n \text{ if } n \text{ is even, and } s_n = \tfrac{1}{2}(n+1) \text{ if } n \text{ is odd};$$

$$\therefore\; \Sigma(-1)^{r+1}r \text{ oscillates infinitely.}$$

If a series is divergent or oscillating, it does not possess a "sum to infinity," as defined above. The phrase, "sum to infinity," is restricted to convergent series.

EXERCISE IV. b

Sum to n terms the following series and find whether they are convergent, divergent or oscillating. Find also the sum to infinity, when it exists.

1. $1 + 3 + 5 + 7 + \dots$. 2. $1 - 3 + 5 - 7 + 9 - \dots$.

3. $(1 - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{7}) + \dots$.

4. $1 + 2 + 1 + 2 + 1 + 2 + \dots$. 5. $1 + 0{\cdot}1 + 0{\cdot}01 + 0{\cdot}001 + \dots$.

6. $1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + \dots$.

7. $1 - 2 + 4 - 8 + 16 - \dots$. 8. $1 - 0{\cdot}1 + 0{\cdot}01 - 0{\cdot}001 + \dots$.

9. $\frac{1}{4} - 1 + \frac{3}{4} + \frac{5}{4} - 3 + \frac{7}{4} + \frac{9}{4} - 5 + \frac{11}{4} + \dots$.

10. $\log 2 + \log \frac{3}{2} + \log \frac{4}{3} + \log \frac{5}{4} + \dots$.

11. $\dfrac{1}{2 \,.\, 5} + \dfrac{1}{5 \,.\, 8} + \dfrac{1}{8 \,.\, 11} + \dots$.

12. $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dfrac{4}{5!} + \dots$.

13. $(x + y) + (x^2 + y^2) + (x^3 + y^3) + \dots$, $x > y > 0$.

14. $(1 - x) + (x - x^2) + (x^2 - x^3) + \dots$
(i) if $0 < x < 1$; (ii) if $x = 1$; (iii) if $x > 1$.

15. $x + \dfrac{x}{1 + x} + \dfrac{x}{(1 + x)^2} + \dots$
(i) if $x > 0$; (ii) if $x = 0$; (iii) if $-1 < x < 0$;
(iv) if $-2 < x < -1$; (v) if $x = -2$; (vi) if $x < -2$.

16. Prove that $\dfrac{1}{1 \,.\, 3} + \dfrac{1}{3 \,.\, 5} + \dots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.
Hence find the sum to infinity of $\dfrac{1}{1 \,.\, 3} + \dfrac{1}{3 \,.\, 5} + \dfrac{1}{5 \,.\, 7} + \dots$.

17. Use the identity $\dfrac{3}{(3r-2)(3r+1)} \equiv \dfrac{1}{3r-2} - \dfrac{1}{3r+1}$ to sum to infinity the series $\dfrac{1}{1 \,.\, 4} + \dfrac{1}{4 \,.\, 7} + \dfrac{1}{7 \,.\, 10} + \dots$.

18. Prove that $\sum_{1}^{n} \frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(n+1)(n+2)}$.

Deduce the sum to infinity of the series,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots .$$

19. Prove that, if $x \neq 1$,

$$1 + 2x + 3x^2 + \dots + nx^{n-1} = \frac{1 - x^{n+1}}{(1-x)^2} - \frac{(n+1)x^n}{1-x}.$$

Deduce the sum to infinity of the series,

$$1 + 2x + 3x^2 + \dots \text{ if } -1 < x < 1.$$

20. Use Example 5, p. 45, to find the sum to infinity of the series, $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$, if $-1 < x < 1$.

21. Sum to n terms $1 + \frac{2.3}{1.2}x + \frac{3.4}{1.2}x^2 + \frac{4.5}{1.2}x^3 + \dots$, and deduce the sum to infinity if $-1 < x < 1$.

If the sum to n terms, s_n, can be expressed by elementary functions, the nature of the series can be determined by finding whether this expression tends to a limit or diverges or oscillates, when $n \to \infty$. But the main object of this chapter is to discuss methods for deciding the question of convergence when it is impossible or inconvenient to express s_n in this way.

It is beyond the scope of this book to give rigorous proofs of those fundamental theorems on limits upon which the principles of convergence depend. The reader will, however, find little difficulty in accepting the results enunciated below.

It is often convenient to use the following symbol:

$|x|$ denotes the value of x if x is positive, and the value of $-x$ if x is negative. It is called the **modulus of x**, or more shortly "mod x."

Thus the *positive* square root of a^2 is denoted by $|a|$, the *difference* between a and b can be written $|a-b|$ without specifying which of the numbers a, b is the greater, also the condition $-1 < x < 1$ can be put in the more concise form, $|x| < 1$.

General Properties of Limits

If $\lim_{n\to\infty} s_n = s$, and if c is any constant, then $\lim_{n\to\infty} (cs_n) = cs$.

If $s_n \to \infty$ when $n \to \infty$ and if c is any *positive* constant, then $cs_n \to \infty$ when $n \to \infty$; if c is any *negative* constant, $cs_n \to -\infty$.

The first theorem states that if $|s_n - s|$ is as small as we please for *all* sufficiently large values of n, so is $|cs_n - cs|$. This is true because $|cs_n - cs|$ is less than any chosen positive number k for all values of n for which $|s_n - s|$ is less than $\frac{k}{c}$, if $c \neq 0$. If $c=0$, both cs_n and cs are zero.

The second theorem states that if s_n takes values as large as we please for *all* sufficiently large values of n, so also does cs_n if c is positive; and if c is negative, cs_n takes negative values, *numerically* as large as we please for *all* sufficiently large values of n.

If $\lim\limits_{n\to\infty} s_n = s$ and if $\lim\limits_{n\to\infty} t_n = t$, then $\lim\limits_{n\to\infty} (as_n + bt_n) = as + bt$, where a, b are constants.

The theorem states that if $|s_n - s|$ and $|t_n - t|$ are both as small as we please for *all* sufficiently large values of n, so also is $|(as_n + bt_n) - (as + bt)|$, because it equals $|a(s_n - s) + b(t_n - t)|$.

For example, since $\lim\limits_{n\to\infty} \frac{3n+2}{n} = 3$ and $\lim\limits_{n\to\infty} (\frac{1}{2})^n = 0$, it follows that

$$\lim_{n\to\infty} \left\{\frac{7(3n+2)}{n} - 1000(\tfrac{1}{2})^n\right\} = (7 \times 3) - 0 = 21.$$

If $\lim\limits_{n\to\infty} s_n = s$ and if $\lim\limits_{n\to\infty} t_n = t$, then

$$\text{(i) } \lim_{n\to\infty} (s_n t_n) = st; \quad \text{(ii) } \lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{s}{t} \text{ if } t \neq 0.$$

(i) can be inferred from the identity,

$$s_n t_n - st \equiv s_n(t_n - t) + t(s_n - s).$$

(ii) can be inferred from the identity,

$$\frac{s_n}{t_n} - \frac{s}{t} \equiv \frac{t(s_n - s) - s(t_n - t)}{t_n t}.$$

Example 1. Find the values of

$$\text{(i) } \lim_{n\to\infty} \frac{(3n+2)(2n-5)}{n^2}; \quad \text{(ii) } \lim_{n\to\infty} \frac{2n^2+n-1}{3n^2-n+5}.$$

(i) $\dfrac{(3n+2)(2n-5)}{n^2} = \left(3 + \dfrac{2}{n}\right)\left(2 - \dfrac{5}{n}\right)$;

$$\therefore \lim_{n\to\infty} \frac{(3n+2)(2n-5)}{n^2} = \lim_{n\to\infty}\left(3 + \frac{2}{n}\right) \cdot \lim_{n\to\infty}\left(2 - \frac{5}{n}\right)$$
$$= 3 \times 2 = 6.$$

(ii) $\dfrac{2n^2+n-1}{3n^2-n+5}=\left(2+\dfrac{1}{n}-\dfrac{1}{n^2}\right)\div\left(3-\dfrac{1}{n}+\dfrac{5}{n^2}\right)$;

$$\therefore \lim_{n\to\infty}\frac{2n^2+n-1}{3n^2-n+5}=\lim_{n\to\infty}\left(2+\frac{1}{n}-\frac{1}{n^2}\right)\div\lim_{n\to\infty}\left(3-\frac{1}{n}+\frac{5}{n^2}\right)$$
$$=2\div 3=\tfrac{2}{3}.$$

If s_n is a function of n which steadily increases as n increases and if s_n is always less than some fixed number c, then $\lim\limits_{n\to\infty} s_n$ exists and has a value less than or equal to c.

We shall not prove this theorem in this book, but it can be illustrated graphically as follows :

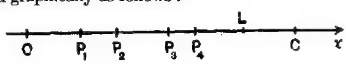

FIG. 1.

Draw a straight line Ox and, following the usual conventions for sign, mark on it a point C such that OC $=c$. Mark points P_1, P_2, P_3, ... on Ox such that

$$OP_1=s_1,\ OP_2=s_2,\ OP_3=s_3,\ \dots .$$

Since s_n *steadily increases* as n increases, the positions of P_1, P_2, P_3, ... advance steadily from left to right along Ox. Since $s_n<c$ for *all* values of n, every point P_n lies to the left of C.

The theorem says that there exists a point L, either on the left of C or coincident with C, beyond which P_n never passes and which is such that the length of P_nL is as small as we please for *all* sufficiently large values of n.

Then, if OL $=l$, $\lim\limits_{n\to\infty} s_n=l$ where $l\leqslant c$.

Example 2. Discuss the behaviour of s_n when $n\to\infty$, if

$$s_n\equiv 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{(n-1)!}.$$

$$s_n=1+1+\frac{1}{2}+\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}+\dots+\frac{1}{2\,.\,3\,.\,4\dots(n-1)};$$

suppose $n>4$, then

$$s_n<1+1+\frac{1}{2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\frac{1}{4\,.\,5}+\dots+\frac{1}{(n-2)(n-1)};$$

$$\therefore\ s_n<1+1+\tfrac{1}{2}+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\dots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right);$$

$\therefore\ s_n < 1 + 1 + \frac{1}{2} + \frac{1}{2} - \dfrac{1}{n-1} = 3 - \dfrac{1}{n-1} < 3.$

But $s_1 < s_2 < s_3 < \ldots < s_n < s_{n+1} < \ldots$.

$\therefore\ s_n$ steadily increases as n increases but is always less than 3.

$\therefore\ s_n$ tends to a limit when $n \to \infty$ and this limit $\leqslant 3$.

The Number e

The value of the limit in Example 2 is not a rational number but, like π, it can be calculated to as many places of decimals as required. It is always denoted by the letter e, and it can be shown that $e = 2{\cdot}71828 \ldots$. Properties of the number e will be discussed in Ch. VI.

General Theorems on Convergence

The properties of limits, discussed above, may be used to obtain some general theorems on convergence.

If $u_1 + u_2 + u_3 + \ldots$ is convergent with sum to infinity s, then $u_{m+1} + u_{m+2} + \ldots$ is convergent and its sum to infinity is

$$s - (u_1 + u_2 + \ldots + u_m),$$

where m is any given positive integer.

$$\begin{aligned}\lim_{n\to\infty} &(u_{m+1} + u_{m+2} + \ldots + u_{m+n}) \\ &= \lim_{n\to\infty} [(u_1 + u_2 + \ldots + u_{m+n}) - (u_1 + u_2 + \ldots + u_m)] \\ &= \lim_{n\to\infty} (u_1 + u_2 + \ldots + u_{m+n}) - (u_1 + u_2 + \ldots + u_m) \\ &\qquad\qquad\qquad = s - (u_1 + u_2 + \ldots + u_m).\end{aligned}$$

Similarly if $u_1 + u_2 + \ldots$ diverges or oscillates, so also does $u_{m+1} + u_{m+2} + \ldots$, where m is any given positive integer.

Therefore in discussing the convergence of a series, any finite number of terms at the beginning may be ignored.

If $u_1 + u_2 + u_3 + \ldots$ is convergent with sum to infinity s, and if c is any constant, then $cu_1 + cu_2 + cu_3 + \ldots$ is convergent with sum to infinity cs.

$$\lim_{n\to\infty} (cu_1 + cu_2 + \ldots + cu_n) = c \lim_{n\to\infty} (u_1 + u_2 + \ldots + u_n) = cs.$$

Similarly if $u_1 + u_2 + \ldots$ diverges or oscillates, so also does $cu_1 + cu_2 + \ldots$, unless $c = 0$.

If $u_1+u_2+\ldots$ and $v_1+v_2+\ldots$ are both convergent, with sums to infinity s and t, then $(au_1+bv_1)+(au_2+bv_2)+\ldots$ is also convergent with sum to infinity as + bt, where a, b are constants.

$$\lim_{n\to\infty}\{(au_1+bv_1)+(au_2+bv_2)+\ldots+(au_n+bv_n)\}=$$
$$a\lim_{n\to\infty}(u_1+u_2+\ldots+u_n)+b\lim_{n\to\infty}(v_1+v_2+\ldots+v_n)=as+bt.$$

If $u_1+u_2+\ldots$ is convergent, $\lim\limits_{n\to\infty} u_n=0$.

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}(u_1+u_2+\ldots+u_n)-\lim_{n\to\infty}(u_1+u_2+\ldots+u_{n-1})$$
$$=s-s=0.$$

The converse theorem is not true; the series $u_1+u_2+\ldots$ may be divergent when $\lim\limits_{n\to\infty}u_n=0$. This is evident from the following example.

Example 3. Prove that the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

is divergent.

$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2};\quad \tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{4}{8}=\tfrac{1}{2};$$

and so on.

$$\therefore\ s_4>1+\tfrac{1}{2}+\tfrac{1}{2};\quad s_8>1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2};\quad s_{16}>1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2};$$

and so on.

$\therefore$ the sum of 2^p terms is greater than $1+\tfrac{1}{2}p$.

$\therefore\ s_n\to\infty$ when $n\to\infty$.

$\therefore$ the series is divergent, although $u_n=\dfrac{1}{n}\to 0$ when $n\to\infty$.

Tests for Convergence

Certain useful tests for convergence apply only to series in which all the terms are positive. These will be discussed first. Such tests can also be used for series which contain only a finite number of negative terms, because the convergence is unaffected if any finite number of terms at the beginning of a series are omitted.

Series of Positive Terms

A series of positive terms cannot oscillate. If a constant k exists such that

$$s_n = u_1 + u_2 + \ldots + u_n < k$$

for all values of n, the series $u_1 + u_2 + \ldots$ is convergent. If no such number k exists, the series is divergent.

Since all the terms are positive, s_n steadily increases as n increases.

$\therefore$ if $s_n < k$ for all values of n, $\lim_{n\to\infty} s_n$ exists and is equal to k or is less than k, and the series is convergent.

If whatever number k is chosen, a value of n exists for which $s_n > k$, then $s_r > k$ whenever $r > n$, and therefore $s_n \to \infty$ when $n \to \infty$.

The series is therefore either convergent or diverges to ∞, and so it cannot oscillate.

Comparison Tests

If $u_1 + u_2 + \ldots$ and $v_1 + v_2 + \ldots$ are two series of positive terms and if Σv_n is convergent, then Σu_n is convergent if either

(i) $u_n \leqslant cv_n$ for all values of n greater than a fixed integer k

or (ii) $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

where c, l are positive constants.

(*i*) If t is the sum to infinity of $v_1 + v_2 + \ldots$,

$$v_1 + v_2 + \ldots + v_n < t \text{ for all values of } n.$$

$$\therefore\ u_{k+1} + u_{k+2} + \ldots + u_n \leqslant c(v_{k+1} + v_{k+2} + \ldots + v_n) < ct$$

for *all* values of n greater than k.

But k is fixed, $\therefore$ $u_1 + u_2 + \ldots$ is convergent.

(ii) Since $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, the value of $\frac{u_n}{v_n}$ is as near l as we please for all sufficiently large values of n.

$\therefore$ a fixed integer k exists such that $\frac{u_n}{v_n} < l + 1$, say, whenever $n > k$.

$\therefore$ it follows from (i) that Σu_n is convergent.

Example 4. Prove that the series

$$1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}+\dots \text{ is convergent.}$$

Compare this with the series

$$1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^2}{2!}\left(\frac{2}{3}\right)+\frac{2^2}{2!}\left(\frac{2}{3}\right)^2+\frac{2^2}{2!}\left(\frac{2}{3}\right)^3+\dots.$$

From the 5th term onwards, each term of the given series Σu_n is less than the corresponding term of the series Σv_n chosen for comparison, that is, $u_n < v_n$ whenever $n > 4$.

But if we omit the first two terms of Σv_n, we have a G.P. with common ratio $\frac{2}{3}$, which is therefore convergent.

$\therefore$ Σv_n is convergent; $\therefore$ Σu_n is convergent.

If $u_1+u_2+\dots$ and $v_1+v_2+\dots$ are two series of positive terms and if Σv_n is divergent, then Σu_n is divergent if either

(i) $u_n \geqslant cv_n$ for all values of n greater than a fixed integer k

or (ii) $\lim\limits_{n\to\infty} \frac{u_n}{v_n}=l$, where c, l are positive constants.

(i) Since Σv_n is divergent and since k is fixed,

$$(v_{k+1}+v_{k+2}+\dots+v_n)$$

takes values as large as we please for *all* sufficiently large values of n.

But $(u_{k+1}+u_{k+2}+\dots+u_n) \geqslant c(v_{k+1}+v_{k+2}+\dots+v_n)$ where $c>0$;

$\therefore$ $(u_{k+1}+u_{k+2}+\dots+u_n)$ takes values as large as we please for all sufficiently large values of n; $\therefore$ Σu_n is divergent.

(ii) Since $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l$, where $l>0$, the value of $\frac{u_n}{v_n}$ is as near l as we please for *all* sufficiently large values of n; $\therefore$ $\frac{u_n}{v_n}>\frac{1}{2}l$, say, since $l\neq 0$, for all sufficiently large values of n. The result then follows from (i).

Example 5. Prove that the series

$$\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\frac{4}{7.9}+\dots \text{ is divergent.}$$

The given series is Σu_n where $u_n=\frac{n}{(2n-1)(2n+1)}$;

$\therefore$ $u_n>\frac{n}{2n\,.\,3n}=\frac{1}{6n}$. Compare Σu_n with Σv_n where $v_n=\frac{1}{n}$.

Since $u_n > \frac{1}{6}v_n$ and since Σv_n is divergent, it follows that Σu_n is divergent.

The utility of the comparison tests depends on having at our disposal standard series which are known to be convergent or divergent. One such standard series is the Geometrical Progression

$$1 + r + r^2 + r^3 + \ldots,$$

which is convergent if $0 \leqslant r < 1$ and is divergent if $r \geqslant 1$. This was employed for comparison in Example 4. We shall now consider another series, a special case of which was discussed on p. 64 and used in Example 5.

The series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

is convergent if $p > 1$ and is divergent if $p \leqslant 1$.

(i) Suppose $p > 1$.

$$s_n = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p}$$

$$< \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \ldots + \frac{1}{7^p}\right) + \ldots \text{ to } n \text{ brackets}$$

where successive brackets contain 1, 2, 4, 8, 16, ... terms.

$$\therefore\ s_n < \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \ldots \text{ to } n \text{ terms};$$

$$\therefore\ s_n < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \ldots \text{ to } n \text{ terms};$$

$$\therefore\ s_n < \frac{1}{1 - (\frac{1}{2})^{p-1}} \text{ if } p > 1;$$

that is, s_n is less than a constant k, for all values of n.

$\therefore$ by p. 65, the series is convergent.

(ii) Suppose $p = 1$.

The series is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, and it was shown on p. 64 that this is divergent.

(iii) Suppose $p < 1$.

If $p < 1$, $n^p < n$ when $n > 1$; $\therefore\ \dfrac{1}{n^p} > \dfrac{1}{n}$ when $n > 1$.

$\therefore$ each term of the series $\Sigma\frac{1}{n^p}$, after the first, is greater than the corresponding term of the series $\Sigma\frac{1}{n}$.

But $\Sigma\frac{1}{n}$ is divergent, $\therefore$ $\Sigma\frac{1}{n^p}$ is divergent if $p<1$.

Example 6. Examine the series

$$\frac{5}{1.2.4}+\frac{7}{2.3.5}+\frac{9}{3.4.6}+\frac{11}{4.5.7}+\ldots.$$

The given series is Σu_n where $u_n=\frac{2n+3}{n(n+1)(n+3)}$.

$$\therefore\ u_n<\frac{3n}{n.n.n}=\frac{3}{n^2},$$

Compare Σu_n with Σv_n where $v_n=\frac{1}{n^2}$.

Since $u_n<3v_n$ and since Σv_n is convergent, it follows that Σu_n is convergent.

EXERCISE IV. c

Determine whether the following series of positive terms are convergent or divergent.

1. $\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots$.

2. $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\ldots$.

3. $\frac{1}{1\cdot1^2}+\frac{1}{1\cdot2^2}+\frac{1}{1\cdot3^2}+\frac{1}{1\cdot4^2}+\ldots$.

4. $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots$.

5. $\frac{1}{2.4}+\frac{2}{4.6}+\frac{3}{6.8}+\ldots$.

6. $\frac{6}{1.3.5}+\frac{8}{3.5.7}+\frac{10}{5.7.9}+\ldots$.

7. $\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\ldots$.

8. $\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{27}}+\frac{1}{\sqrt{64}}+\frac{1}{\sqrt{125}}+\ldots$.

9. $\Sigma\frac{n-1}{2n^3+1}$.

10. $\Sigma\frac{10n+1}{n^3}$.

11. $\Sigma\frac{1}{\sqrt{n}}$.

12. $\Sigma\frac{10^n}{n!}$.

13. $\Sigma\frac{\sqrt{n}}{n+1}$.

14. $\Sigma\frac{\sqrt{n}-1}{n^2+1}$.

15. $\Sigma\frac{3^n+1}{4^n+1}$.

16. $\Sigma\frac{n}{n^3-10}$.

17. $\Sigma\frac{(n+2)(n+4)}{n(n+3)(n+5)}$.

18. $\dfrac{1}{1+a}+\dfrac{a}{1+a^2}+\dfrac{a^2}{1+a^3}+\dots$ where $a>0$.

19. $\dfrac{1}{1+a}+\dfrac{1}{1+a^2}+\dfrac{1}{1+a^3}+\dots$ where $a>0$.

20. $\dfrac{1}{1+1}+\dfrac{a}{1+a^2}+\dfrac{a^2}{1+a^4}+\dfrac{a^3}{1+a^6}+\dots$ where $a>0$.

21. $\dfrac{1}{a+d}+\dfrac{1}{a+2d}+\dfrac{1}{a+3d}+\dots$ where $a>0$, $d>0$.

22. $\Sigma\dfrac{1}{\sqrt{(n^2+2n)}}$. 23. $\Sigma\dfrac{1}{\sqrt{(n+1)}-\sqrt{n}}$.

24. $\Sigma\dfrac{n^p}{\sqrt{(n+1)}-\sqrt{n}}$.

D'Alembert's Ratio Test

The series of positive terms, $u_1+u_2+\dots$, is convergent if either

(i) $\dfrac{u_n}{u_{n+1}}>k>1$ whenever $n\geqslant m$ where m is a fixed integer and k is a constant,

or (ii) $\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}=l>1$.

(i) $u_m>ku_{m+1}$; $\therefore\ u_{m+1}<\dfrac{u_m}{k}$;

similarly $u_{m+2}<\dfrac{u_{m+1}}{k}<\dfrac{u_m}{k^2}$; $u_{m+3}<\dfrac{u_{m+2}}{k}<\dfrac{u_m}{k^3}$; and so on.

$\therefore$ each term of the series $u_{m+1}+u_{m+2}+u_{m+3}+\dots$ is less than the corresponding term of the G.P.,

$$\frac{u_m}{k}+\frac{u_m}{k^2}+\frac{u_m}{k^3}+\dots .$$

Since the common ratio $\dfrac{1}{k}$ is less than 1, this G.P. is convergent.

$\therefore$ since m is fixed, Σu_n is convergent.

(ii) Since $\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}=l>1$, the value of $\dfrac{u_n}{u_{n+1}}$ is as near l as we please for all sufficiently large values of n. $\therefore$ if k is a constant such that $l>k>1$, we have $\dfrac{u_n}{u_{n+1}}>k>1$ for all sufficiently large values of n. The result then follows from (i).

The series of positive terms, $u_1 + u_2 + \dots$, is divergent if either

(i) $\dfrac{u_n}{u_{n+1}} \leqslant 1$ whenever $n \geqslant m$ where m is a fixed integer,

or (ii) $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = l < 1$.

(i) $u_{m+1} \geqslant u_m$; $u_{m+2} \geqslant u_{m+1} \geqslant u_m$; $u_{m+3} \geqslant u_{m+2} \geqslant u_m$; and so on.

$\therefore$ each term of the series, $u_{m+1} + u_{m+2} + u_{m+3} + \dots$ is greater than or equal to the corresponding term of the divergent series,

$$u_m + u_m + u_m + \dots .$$

$\therefore$ since m is fixed, Σu_n is divergent.

(ii) Since $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = l < 1$, the value of $\dfrac{u_n}{u_{n+1}}$ is as near l as we please for all sufficiently large values of n. But $l < 1$, $\therefore \dfrac{u_n}{u_{n+1}} \leqslant 1$ for all sufficiently large values of n. The result tnerefore follows from (i).

Each of these limit-tests is useless if $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = 1$. In this case the series may be either convergent or divergent.

Thus if $u_n = \dfrac{1}{n}$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$;

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1.$$

And if $v_n = \dfrac{1}{n^2}$, $\quad \dfrac{v_n}{v_{n+1}} = \dfrac{(n+1)^2}{n^2} = 1 + \dfrac{2}{n} + \dfrac{1}{n^2}$;

$$\therefore \lim_{n\to\infty} \frac{v_n}{v_{n+1}} = 1.$$

But $\Sigma \dfrac{1}{n}$ is divergent and $\Sigma \dfrac{1}{n^2}$ is convergent.

Further, *no conclusion can be drawn from the non-existence of the limit of* $\dfrac{u_n}{u_{n+1}}$.

Thus for the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{2^3} + \frac{1}{4^3} + \dots$$

successive values of $\dfrac{u_n}{u_{n+1}}$ are 2, 1, 4, $\frac{1}{2}$, 8, $\frac{1}{4}$, 16, $\frac{1}{8}$, ... , so that $\dfrac{u_n}{u_{n+1}}$ oscillates infinitely, and yet the series is convergent.

Example 7. Prove that the series

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots$$

is convergent for $0 \leqslant x < 1$ and divergent for $x \geqslant 1$.

$$u_n=\frac{x^n}{n},\ u_{n+1}=\frac{x^{n+1}}{n+1};\ \therefore\ \text{if } x\neq 0,\ \frac{u_n}{u_{n+1}}=\frac{n+1}{nx}=\left(1+\frac{1}{n}\right)\frac{1}{x};$$

$$\therefore\ \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}.$$

If $0<x<1$, $\frac{1}{x}>1$, $\therefore$ Σu_n is convergent.

If $x>1$, $\frac{1}{x}<1$, $\therefore$ Σu_n is divergent.

If $x=1$, the test fails; the series then becomes

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\dots,$$

which is divergent, see p. 64.

If $x=0$, each term is 0, and the series is convergent.

EXERCISE IV. d

Discuss the behaviour of the following series, assuming that x is positive.

1. $1+2x+3x^2+4x^3+\dots$.

2. $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$.

3. $\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\dots$.

4. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\dots$.

5. $\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\dots$.

6. $\frac{1}{x}+\frac{2!}{x^2}+\frac{3!}{x^3}+\dots$.

7. $x+2^2x^2+3^2x^3+4^2x^4+\dots$.

8. $x+\frac{1}{x}+x^2+\frac{1}{x^2}+x^3+\frac{1}{x^3}+\dots$.

9. $1+(2+3)x+(2^2+3^2)x^2+(2^3+3^3)x^3+\dots$.

10. $\frac{1}{2}x+\frac{1.3}{2.5}x^2+\frac{1.3.5}{2.5.8}x^3+\frac{1.3.5.7}{2.5.8.11}x^4+\dots$.

11. $\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\frac{1.3.5.7}{2.4.6.8}x^4+\dots$.

12. $1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\dots$.

13. $x+2^4\frac{x^2}{2!}+3^4\frac{x^3}{3!}+4^4\frac{x^4}{4!}+\dots$.

14. $\dfrac{1}{x+1}+\dfrac{x}{x+2}+\dfrac{x^2}{x+3}+\dfrac{x^3}{x+4}+\dots$.

15. $\dfrac{1}{x+1}+\dfrac{1}{2x^2+1}+\dfrac{1}{3x^3+1}+\dots$.

Series of Positive and Negative Terms

The proof, given on p. 64, that $\lim\limits_{n\to\infty} u_n = 0$ is a *necessary* condition for convergence still holds good when the series contains both positive and negative terms, and this fact sometimes makes it possible to say that a given series cannot be convergent. But here again it must be remembered that the converse is not true, and it is easy to construct oscillating and divergent series for which $\lim\limits_{n\to\infty} u_n = 0$. Thus

$$1-\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{5}+\dots$$

oscillates finitely, and

$$1-\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{5}+\tfrac{1}{5}+\tfrac{1}{5}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}+\dots$$

diverges.

The most useful method is that of "absolute" convergence, which is explained on p. 74. But series whose terms are alternately positive and negative can often be dealt with by the following theorem :

If $u_1-u_2+u_3-u_4+\dots$ is a series of terms alternately positive and negative, and if $u_n > u_{n+1}$ for all values of n, and if $\lim\limits_{n\to\infty} u_n = 0$, then the series is convergent.

$$s_{2n}=(u_1-u_2)+(u_3-u_4)+\dots+(u_{2n-1}-u_{2n}).$$

Since each bracket is positive, s_{2n} steadily increases as n increases, that is, $s_2 < s_4 < s_6 < s_8 < \dots$.

But $s_{2n}=u_1-(u_2-u_3)-(u_4-u_5)-\dots-(u_{2n-2}-u_{2n-1})-u_{2n}$.

Since each bracket is positive, $s_{2n} < u_1$.

$\therefore \lim\limits_{n\to\infty} s_{2n}$ exists and equals l, say, where $l \leqslant u_1$.

But $s_{2n+1}=s_{2n}+u_{2n+1}$ and $\lim\limits_{n\to\infty} u_{2n+1}=0$,

$$\therefore \lim_{n\to\infty} s_{2n+1} \text{ exists and } = \lim_{n\to\infty} s_{2n} = l.$$

$$\therefore \text{ the series is convergent.}$$

Example 8. Prove that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$ is convergent.

In this series,

(i) the terms are *alternately* positive and negative ;

(ii) the terms steadily decrease in numerical value ;

(iii) $\lim\limits_{n\to\infty} u_n=0$ since $\lim\limits_{n\to\infty}\frac{1}{n}=0.$

$\therefore$ the *three* conditions for convergence are all satisfied.

It is important to notice that if any one of the *three* restrictions imposed by the data of this theorem is removed, the series need not be convergent.

Thus in the series $1\frac{1}{2}-1\frac{1}{4}+1\frac{1}{8}-1\frac{1}{16}+\ldots$

$$s_{2n}=\tfrac{1}{3}[1-(\tfrac{1}{2})^{2n}] \text{ and } s_{2n+1}=1+\tfrac{1}{3}[1+(\tfrac{1}{2})^{2n+1}].$$

$\therefore$ when $n\to\infty$, $s_{2n}\to\frac{1}{3}$ and $s_{2n+1}\to 1\frac{1}{3}$.

$\therefore$ the series oscillates finitely and is not convergent. Here, the terms are alternately positive and negative, and steadily decrease in numerical value, but $\lim\limits_{n\to\infty} u_n\neq 0$.

Again, since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is divergent and since

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

is convergent, the series

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{16}+\ldots$$

is divergent.

Here, the terms are alternately positive and negative, and $\lim\limits_{n\to\infty} u_n=0$, but the terms do not steadily decrease in numerical value.

Again, since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is divergent and since

$$\frac{2}{2}+\frac{2}{4}+\frac{2}{8}+\frac{2}{16}+\ldots$$

is convergent, the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\ldots+\frac{1}{15}-\frac{1}{16}+\frac{1}{17}+\ldots$$

is divergent.

Here, the terms steadily decrease in numerical value and $\lim\limits_{n\to\infty} u_n=0$, but the terms are not *alternately* positive and negative.

Absolute Convergence

If a series of positive terms is convergent, and if a new series is obtained by changing the signs of any of the terms of the given series, then the new series is convergent.

Suppose $u_1+u_2+u_3+u_4+\dots$ is a convergent series of positive terms, and suppose the new series is

$$u_1+u_2-u_3+u_4-u_5-u_6-u_7+u_8+u_9-u_{10}+\dots.$$

Consider the two following series,

$$u_1+u_2+0+u_4+0+0+0+u_8+u_9+0+\dots,$$
$$0+0+u_3+0+u_5+u_6+u_7+0+0+u_{10}+\dots.$$

Compare each of these with the given convergent series,

$$u_1+u_2+u_3+u_4+u_5+\dots.$$

From the test on p. 65, taking $c=1$, it follows that each of these two series is convergent.

$\therefore$ by p. 64, the "difference" of these two series

$$u_1+u_2-u_3+u_4-u_5-u_6-u_7+u_8+\dots$$

is also convergent.

The same argument can obviously be used whichever terms of the original series have their signs changed.

If a series contains positive and negative terms, and if the new series obtained by changing the sign of every negative term is convergent, the original series is called **absolutely convergent.**

Thus the series,

$$1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}-\dots,$$

is called "absolutely convergent" because the series,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots,$$

is known to be convergent (see p. 67).

This definition does not state that an absolutely convergent series Σu_n is itself convergent, but only that a *different* series, namely $\Sigma|u_n|$ is convergent. The phrase "absolutely convergent" would of course be most misleading if a series could be "absolutely convergent" but not "convergent," that is to

say, if $\Sigma|u_n|$ could be convergent when Σu_n was not convergent. But it has just been proved that Σu_n must be convergent whenever $\Sigma|u_n|$ is convergent. This fact is often stated in the form,

A series which is absolutely convergent is also convergent.

Thus the series,

$$1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}-\dots,$$

is absolutely convergent and must therefore also be convergent.

If a series of positive and negative terms is convergent, and if the new series obtained by changing the sign of every negative term is divergent, the original series is called **conditionally convergent**, or sometimes **semi-convergent**.

Thus the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$ is convergent (p. 73),

but the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$ is divergent (p. 64).

$\therefore$ the series, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$, is called conditionally convergent.

Example 9. Prove that the series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$$

is convergent for all values of x.

(i) Suppose $x>0$.

All the terms are positive; $\therefore$ we can use D'Alembert's test, p. 69,

$$u_n=\frac{x^{n-1}}{(n-1)!},\quad u_{n+1}=\frac{x^n}{n!};\quad \therefore\ \frac{u_n}{u_{n+1}}=\frac{n}{x}.$$

$\therefore\ u_n>2u_{n+1}$ whenever $n>2x$.

$\therefore$ the series is convergent for all positive values of x.

(ii) Suppose $x<0$.

Put $x=-y$ where $y>0$. The series becomes

$$1-y+\frac{y^2}{2!}-\frac{y^3}{3!}+\frac{y^4}{4!}-\dots.$$

From (i), it follows that this series is absolutely convergent for all values of y; it is therefore also convergent. Therefore the given series is convergent for all negative values of x.

(iii) If $x=0$, the series is $1+0+0+0+\dots$ and is therefore convergent.

Note. Since this series is convergent, it follows from p. 64 that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all values of x. This result can be deduced independently from the relation $u_{n+1} < \frac{1}{2}u_n$ whenever $n > 2x > 0$.

Example 10. Prove that the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

is convergent if $-1 < x \leqslant 1$ and diverges to $-\infty$ if $x \leqslant -1$ and oscillates infinitely if $x > 1$.

Example 7, p. 71, shows that the series is absolutely convergent if $-1 < x < 1$ and is therefore also convergent.

If $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$, which is convergent, p. 73.

If $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots$, which diverges to $-\infty$, p. 64.

If $x < -1$, each term of the series is negative and is *numerically* greater than the corresponding term of the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \dots$, and therefore diverges to $-\infty$.

If $x > 1$, then $x = 1 + c$ where $c > 0$.

Then $x^n = (1+c)^n > 1 + nc + \frac{1}{2}n(n-1)c^2 > \frac{1}{2}n(n-1)c^2$.

$$\therefore \frac{x^n}{n} > \tfrac{1}{2}(n-1)c^2; \quad \therefore \frac{x^n}{n} \to \infty \text{ when } n \to \infty.$$

Also $x^n/n > x^{n-1}/(n-1)$ if $x(n-1) > n$, i.e. if $n > x/(x-1), = m$, say.

$$s_{2n+1} = x + \left(\frac{x^3}{3} - \frac{x^2}{2}\right) + \left(\frac{x^5}{5} - \frac{x^4}{4}\right) + \dots\dots + \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{2n}}{2n}\right);$$

$\therefore$ s_{2n+1} steadily increases as n increases, if $2n > m$;

$\therefore$ either $s_{2n+1} \to \infty$ or $s_{2n+1} \to l$, say, when $n \to \infty$.

$$s_{2n} = -\left(\frac{x^2}{2} - x\right) - \left(\frac{x^4}{4} - \frac{x^3}{3}\right) - \dots\dots - \left(\frac{x^{2n}}{2n} - \frac{x^{2n-1}}{2n-1}\right);$$

$\therefore$ s_{2n} steadily decreases as n increases, if $2n > m$;

$\therefore$ either $s_{2n} \to -\infty$ or $s_{2n} \to l'$, say, when $n \to \infty$.

But $s_{2n+1} - s_{2n} = x^{2n+1}/(2n+1)$. $\therefore$ $s_{2n+1} - s_{2n} \to \infty$ when $n \to \infty$;

$\therefore$ the statements, $s_{2n+1} \to l$, $s_{2n} \to l'$, cannot both be true.

$\therefore$ the series oscillates infinitely.

EXERCISE IV. e

[In this exercise, both positive and negative values of x should be considered.]

Discuss the behaviour of the following series :

1. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$.

2. $\frac{1}{2.3}-\frac{1}{3.4}+\frac{1}{4.5}-\ldots$.

3. $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\ldots$.

4. $1-\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{3}-\frac{1}{4}\sqrt{4}+\ldots$.

5. $1-\frac{1}{2!}+\frac{1}{3}-\frac{1}{4!}+\frac{1}{5}-\frac{1}{6!}+\frac{1}{7}-\ldots$.

6. $1-2x+3x^2-4x^3+\ldots$.

7. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$.

8. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\ldots$.

9. $\frac{x}{2^3}+\frac{x^2}{3^3}+\frac{x^3}{4^3}+\ldots$.

10. $1+(1+2)x+(1+2^2)x^2+(1+2^3)x^3+\ldots$.

11. $1-x+\frac{x^2}{\sqrt{2}}-\frac{x^3}{\sqrt{3}}+\frac{x^4}{\sqrt{4}}-\ldots$.

12. $\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\frac{1}{x+4}+\ldots$, x not integral.

13. $\Sigma\frac{(n+1)x^n}{n^2}$.

14. $\Sigma\frac{n^2-1}{n^2+1}x^n$.

15. $\Sigma\frac{x^n}{1+na}$, $1+na\neq 0$.

16. $\Sigma\frac{2^n-2}{2^n+1}x^n$.

17. $1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots$ where m is not a positive integer and $|x|\neq 1$.

18. $\frac{1}{1.2}+\frac{1}{2.3}-\frac{1}{3.4}-\frac{1}{4.5}+\frac{1}{5.6}+\frac{1}{6.7}-\ldots$.

19. If s_n is the sum of n terms of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, and if t_n is the sum of n terms of $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\ldots$, prove that $t_{3n}=\frac{1}{2}s_{2n}$. What inference can be drawn from this relation ?

20. If s_n is the sum of n terms of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, and if t_n is the sum of n terms of $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\ldots$, prove that $t_{3n}=s_{4n}+\frac{1}{2}s_{2n}$. What inference can be drawn from this relation ?

CHAPTER V

THE BINOMIAL SERIES

If m is **a positive integer,** the series

$$1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\ldots$$

terminates automatically after $(m+1)$ terms, since each later coefficient contains the factor $(m-m)=0$; and the sum of this *finite* series is $(1+x)^m$, (see Ch. II, p. 22).

If m is **a positive fraction or any negative number,** the series does not terminate. For example, the series becomes,

if $m=-1$, $1-x+x^2-x^3+x^4-\ldots$;

if $m=-2$, $1-2x+3x^2-4x^3+\ldots$;

if $m=-\frac{1}{2}$, $1-\frac{1}{2}x+\frac{1.3}{2.4}x^2-\frac{1.3.5}{2.4.6}x^3+\ldots$;

if $m=\frac{1}{2}$, $1+\frac{1}{2}x+\frac{(-1).1}{2.4}x^2-\frac{(-1).1.3}{2.4.6}x^3+\frac{(-1).1.3.5}{2.4.6.8}x^4-\ldots$.

Series of this form are called **binomial series**; they do not possess "sums to infinity" for *all* values of x, but it is easy to prove that a sum to infinity exists if $-1<x<1$.

The binomial series

$$\mathbf{1+mx+\frac{m(m-1)}{1.2}x^2+\ldots+\frac{m(m-1)(m-2)\ldots(m-r+1)}{1.2.3\ldots r}x^r+\ldots}$$

is absolutely convergent if $-1<x<1$.

Denote the rth term by u_r; we may suppose $x\neq 0$, because if $x=0$, the series reduces to $1+0+0+0+\ldots$.

If $x\neq 0$, $\frac{u_r}{u_{r+1}}=\frac{r}{m-r+1}\cdot\frac{1}{x}=-\frac{1}{x}\div\left(1-\frac{m+1}{r}\right)$;

$\therefore \lim_{r\to\infty}\frac{|u_r|}{|u_{r+1}|}=\frac{1}{|x|}>1$ since $|x|<1$.

$\therefore$ by p. 69, the series is absolutely convergent, and therefore convergent (p. 75), for $-1 < x < 1$.

Since $\lim\limits_{r\to\infty} \dfrac{|u_r|}{|u_{r+1}|} < 1$ if $|x| > 1$, the series either diverges or oscillates infinitely for $|x| > 1$. The special case $|x| = 1$, when this ratio test fails, is too difficult for discussion here.

Signs of Terms in the Binomial Series

$$\frac{u_{r+1}}{u_r} = \frac{m-r+1}{r} \cdot x = -\left(1 - \frac{m+1}{r}\right)x.$$

$\therefore$ as soon as $r > m+1$, $\dfrac{u_{r+1}}{u_r}$ is negative if $x > 0$ and is positive if $x < 0$.

$\therefore$ after a certain stage, the terms are alternately positive and negative if $x > 0$, and are all of the same sign if $x < 0$.

Numerical Values of Terms in the Binomial Series

The *numerical* value of any term is not affected by changing x into $-x$: we may therefore take x positive.

The case when m is a *positive integer* has been discussed on p. 26. Two other cases require examination.

(i) Suppose m is a *positive fraction.*

As on p. 26, taking $x > 0$,

$$|u_{r+1}| > |u_r| \text{ so long as } r < \frac{(m+1)x}{1+x}$$

and

$$|u_{r+1}| < |u_r| \text{ whenever } r > \frac{(m+1)x}{1+x}.$$

The position of the numerically greatest term, or of two numerically equal greatest terms, is therefore given by the conditions on p. 26.

(ii) Suppose m is *negative.*

Put $m = -n$ where $n > 0$,

then

$$\frac{u_{r+1}}{u_r} = \frac{-n-r+1}{r} \cdot x = -\frac{n+r-1}{r}x.$$

$\therefore$ taking $x > 0$, $|u_{r+1}| > |u_r|$ if $(n+r-1)x > r$,
or if $(n-1)x > r(1-x)$,

and, if $1-x$ is positive, this requires $r < \dfrac{(n-1)x}{1-x}$ when $0 < x < 1$. If $n \leqslant 1$, there is no positive integral value of $r < \dfrac{(n-1)x}{1-x}$, where

$0 < x < 1$, and therefore the first term is numerically the greatest, and thereafter the terms *steadily decrease* in numerical value.

If for $0 < x < 1$, $\frac{(n-1)x}{1-x}$ = a positive integer k + a positive proper fraction, the argument of p. 26 shows that the terms *steadily increase* in numerical value up to u_{k+1} and thereafter *steadily decrease*; also if $\frac{(n-1)x}{1-x}$ = a positive integer k, there are two numerically equal greatest terms, u_k and u_{k+1}.

Thus for all values of m, if $|x| < 1$, the numerical values of the terms of the binomial series in powers of x *either* steadily decrease from the start *or* steadily increase up to a certain stage and thereafter steadily decrease.

Notation

It is often convenient to write the binomial series in the form

$$m_0 + m_1x + m_2x^2 + \ldots + m_rx^r + \ldots,$$

where $m_0 = 1$ and $m_r = \frac{m(m-1)(m-2)\ldots(m-r+1)}{1.2.3\ldots r}$, $r \neq 0$.

The numbers m_0, m_1, m_2, m_3, ... are called *binomial coefficients*; this name was used in Ch. II (see p. 29), in the special case where m is a positive integer.

Some of the relations between the binomial coefficients, which were established in Ch. I-II for positive integral values of m, are true for all values of m. Thus it may be proved (see p. 13) that, for all values of m,

$$m_r + m_{r-1} = (m+1)_r$$

where r is any positive integer.

If $|\mathbf{x}| < 1$, $\lim\limits_{r\to\infty} \mathbf{m_r x^r} = 0$.

By p. 64, if $u_1 + u_2 + u_3 + \ldots$ is convergent, $\lim\limits_{r\to\infty} u_r = 0$.

But Σm_rx^r is convergent when $|x| < 1$.

$$\therefore \lim_{r\to\infty} m_rx^r = 0 \text{ if } |x| < 1.$$

Sum to Infinity

Expressions for the sum to infinity of some special cases of the binomial series have been obtained in Ch. IV.

Thus the G.P., $1-x+x^2-x^3+\dots$, $|x|<1$, given by $m=-1$, has sum to infinity $\dfrac{1}{1+x}\equiv(1+x)^{-1}$.

Also the series (see Ex. IV. b, No. 19, p. 60),

$$1-2x+3x^2-4x^3+\dots, \quad |x|<1,$$

given by $m=-2$, has sum to infinity $\dfrac{1}{(1+x)^2}\equiv(1+x)^{-2}$.

The general theorem is best proved by using the methods of the Calculus. We shall assume the fact that if $f(x)$ is positive and continuous whenever $a<x<b$, then $\int_a^b f(x)\,dx>0$, and it follows that if $\phi(x)$, $\psi(x)$ are continuous functions such that $\phi(x)>\psi(x)$ whenever $a<x<b$, then

$$\int_a^b \phi(x)\,dx>\int_a^b \psi(x)\,dx, \quad \text{since} \quad \int_a^b \{\phi(x)-\psi(x)\}\,dx>0.$$

The meaning of this statement can be illustrated in a simple way by regarding the integral as the measure of an area.

According to the usual convention, the function $(1+x)^m$ where $1+x>0$ has *two* values if $m=\dfrac{p}{q}$ where p and q are positive or negative integers, and q is even.

Thus $\quad (1+\frac{1}{3})^{-\frac{1}{2}}=(\frac{4}{3})^{-\frac{1}{2}}=\pm\frac{1}{2}\sqrt{3}$.

But the sum to infinity of the binomial series for any given value of x for which it is convergent has only one value. *In the proof which follows, we shall for simplicity use the symbol* $(1+x)^m$ *to denote the* **positive** *value of* $(1+x)^m$.

Even if it is necessary or desirable to postpone the formal proof of the Binomial Theorem, applications can be practised without difficulty.

The Binomial Theorem for a Fractional or Negative Index

If m is a positive fraction or any negative number, and if $|\mathbf{x}|<1$, the sum to infinity of

$$1+\mathbf{mx}+\frac{\mathbf{m(m-1)}}{1\,.\,2}\mathbf{x}^2+\frac{\mathbf{m(m-1)(m-2)}}{1\,.\,2\,.\,3}\mathbf{x}^3+\dots$$

is the POSITIVE value of $(1+\mathbf{x})^{\mathbf{m}}$.

Denote the sum to n terms by $f_n(x)$.

Then $f_n(t) = m_0 + m_1 t + m_2 t^2 + \dots + m_r t^r + \dots + m_{n-1} t^{n-1}$;

$$\therefore \frac{d}{dt} f_n(t) = m_1 + 2m_2 t + \dots + r m_r t^{r-1} + \dots + (n-1) m_{n-1} t^{n-2}.$$

Consider the function, $(1+t)\frac{d}{dt} f_n(t) - m f_n(t)$, in which the coefficient of t^r for $r = 0, 1, 2, 3, \dots, n-2$, is

$$(r+1) m_{r+1} + r \,.\, m_r - m \,.\, m_r$$

$$= (r+1)\frac{m(m-1)\dots(m-r+1)(m-r)}{1\,.\,2\dots r(r+1)}$$

$$- (m-r)\frac{m(m-1)\dots(m-r+1)}{1\,.\,2\dots r} = 0;$$

also the coefficient of t^{n-1} is

$$(n-1) m_{n-1} - m \,.\, m_{n-1} = -(m-n+1) m_{n-1} = -n \,.\, m_n.$$

$$\therefore (1+t)\frac{d}{dt} f_n(t) - m f_n(t) = -n \,.\, m_n t^{n-1}.$$

Dividing each side by $(1+t)^{m+1}$, we have

$$\frac{d}{dt}\left\{\frac{f_n(t)}{(1+t)^m}\right\} = -\frac{n \,.\, m_n t^{n-1}}{(1+t)^{m+1}}.$$

Now if t lies between 0 and x where $x > -1$,

$$\frac{1}{(1+t)^{m+1}} \text{ lies between 1 and } \frac{1}{(1+x)^{m+1}}.$$

$$\therefore \frac{d}{dt}\left\{\frac{f_n(t)}{(1+t)^m}\right\} \text{ lies between } -n m_n t^{n-1} \text{ and } -\frac{n m_n}{(1+x)^{m+1}} t^{n-1}.$$

Integrate from $t = 0$ to $t = x$ where $x > -1$. When $t = 0$, $f_n(t) = 1$ and $(1+t)^m = +1$;

$$\therefore \frac{f_n(x)}{(1+x)^m} - 1 = -n m_n \mathsf{A}\int_0^x t^{n-1}\,dt = -\mathsf{A} m_n x^n,$$

where A has some value between 1 and $\dfrac{1}{(1+x)^{m+1}}$.

But from p. 80, when $n \to \infty$, $m_n x^n \to 0$ if $|x| < 1$;

$$\therefore \text{ when } n \to \infty, \frac{f_n(x)}{(1+x)^m} - 1 \to 0 \text{ if } |x| < 1.$$

$$\therefore \lim_{n\to\infty} f_n(x) = (1+x)^m \text{ if } |x| < 1.$$

The reader should notice where precisely this proof breaks down when $x = 1$ and when $x = -1$. It can, however, be shown that the theorem holds for $x = 1$ if $m > -1$, and for $x = -1$ if $m > 0$.

Expansions in Powers of x

If a series $$a_0 + a_1 x + a_2 x^2 + \ldots$$
is convergent when $|x| < k$, say, and if its sum to infinity is $f(x)$, we write $$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots, \quad |x| < k,$$
and the series is called the **expansion** of $f(x)$ in ascending powers of x.

Thus the binomial series,
$$1 + m_1 x + m_2 x^2 + \ldots, \quad |x| < 1,$$
is called the expansion of $(1+x)^m$.

Expansion of $(a+x)^m$

If $m = \frac{p}{q}$ where q is even and if $a + x > 0$, the function $(a+x)^m$ has two values; but for simplicity we shall use the symbol $(a+x)^m$ to denote only the positive value.

If $\left|\frac{x}{a}\right| < 1$, and if a^m exists, we write
$$(a+x)^m = a^m\left(1 + \frac{x}{a}\right)^m = a^m\left\{1 + m_1\frac{x}{a} + m_2\frac{x^2}{a^2} + \ldots\right\}.$$

If $\left|\frac{x}{a}\right| > 1$, we cannot expand the function in ascending powers of x, but, if x^m exists, we may write
$$(a+x)^m = x^m\left(1 + \frac{a}{x}\right)^m = x^m\left\{1 + m_1\frac{a}{x} + m_2\frac{a^2}{x^2} + \ldots\right\},$$
thus expanding it in descending powers, the indices being fractional if m is fractional.

Example 1. Expand $(1-x)^{-3}$ given $|x| < 1$. Find also which is the greatest term in the expansion, if (i) $x = \frac{4}{5}$, (ii) $x = \frac{5}{8}$.

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{1.2}(-x)^2$$
$$+ \frac{(-3)(-4)(-5)}{1.2.3}(-x)^3 + \ldots$$
$$= 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{4.5}{1.2}x^3 + \ldots.$$

The $(r+1)$th term $=\dfrac{(-3)(-4)(-5)\dots(-r-2)}{1\,.\,2\,.\,3\dots r}(-x)^r$

$$=\frac{(r+1)(r+2)}{1\,.\,2}x^r.$$

$$\therefore\ \frac{u_{r+1}}{u_r}=\frac{r+2}{r}\,.\,x.$$

(i) $u_{r+1}>u_r$ so long as $\dfrac{4(r+2)}{5r}>1$ if $x=\frac{4}{5}$, or so long as

$4r+8>5r$, that is, if $r<8$.

If $r=8$, $u_{r+1}=u_r$; and if $r>8$, $u_{r+1}<u_r$.

$\therefore$ the 8th and 9th terms are equal to each other and are greater than all other terms.

(ii) $u_{r+1}>u_r$ so long as $\dfrac{5(r+2)}{8r}>1$ if $x=\frac{5}{8}$, or so long as

$5r+10>8r$, that is, if $r<3\frac{1}{3}$.

If $r\leqslant 3$, $u_{r+1}>u_r$, but if $r>3$, $u_{r+1}<u_r$.

$\therefore$ the 4th term is greater than all other terms.

Example 2. Expand $(1+2x)^{\frac{5}{3}}$ given $|x|<\frac{1}{2}$.

The function can be expanded in ascending powers of x if $|2x|<1$, that is, if $|x|<\frac{1}{2}$. The first 4 terms are

$$1+\tfrac{5}{3}(2x)+\frac{\frac{5}{3}\cdot\frac{2}{3}}{1\,.\,2}(2x)^2+\frac{\frac{5}{3}\cdot\frac{2}{3}\cdot\left(-\frac{1}{3}\right)}{1\,.\,2\,.\,3}(2x)^3$$

$$\text{or}\quad 1+\frac{5}{1}\left(\frac{2x}{3}\right)+\frac{5\,.\,2}{1\,.\,2}\left(\frac{2x}{3}\right)^2+\frac{5\,.\,2\,.\,(-1)}{1\,.\,2\,.\,3}\left(\frac{2x}{3}\right)^3.$$

The terms which follow are alternately positive and negative, and the general term is

$$(-1)^r\frac{5\,.\,2\,.\,1\,.\,4\,.\,7\dots(3r-8)}{1\,.\,2\,.\,3\,.\,4\,.\,5\dots r}\left(\frac{2x}{3}\right)^r,\quad r>2.$$

Example 3. Expand $\dfrac{1}{\sqrt{(1+x^2)}}$ given $x>1$.

$$\frac{1}{\sqrt{(1+x^2)}}=\frac{1}{x}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}};\ \text{but } 0<\frac{1}{x}<1,$$

$$\therefore\ \frac{1}{\sqrt{(1+x^2)}}=\frac{1}{x}\left\{1-\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{x^4}-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{x^6}+\dots\right\}$$

$$=\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x^3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{x^5}-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{x^7}+\dots\,.$$

EXERCISE V. a

Write down the first 4 terms and the coefficient of x^r in the expansions in ascending powers of x of the following functions. In each case, state when the expansion is valid.

1. $(1-x)^{-2}$.
2. $(1+x)^{-3}$.
3. $(1-3x)^{-\frac{1}{2}}$.
4. $(1-5x)^{-1}$.
5. $(1-2x)^{-4}$.
6. $(1+4x)^{\frac{1}{2}}$.
7. $(1-x^2)^{-\frac{2}{3}}$.
8. $(1+3x)^{\frac{2}{3}}$.
9. $\sqrt{(4+x^2)}$.
10. $(1-x)^{-n}$.
11. $\dfrac{x}{1+x^2}$.
12. $(a+x)^{-2}$.
13. $\dfrac{x}{\sqrt{(a^2-x^2)}}$.
14. $\sqrt[3]{(8-x^3)^2}$.
15. $(1-nx)^{-\frac{1}{n}}$.

Find the named terms in the expansions of the following:

16. 10th term in $(1-4x)^{-3}$ if $|x|<\frac{1}{4}$.

17. 9th term in $(1-5x)^{-2}$ if $|x|>\frac{1}{5}$.

18. 4th term in $(1+3x)^{2\frac{1}{2}}$ if $|x|<\frac{1}{3}$.

19. 5th term in $(1+4x^2)^{-2\frac{1}{2}}$ if $|x|>\frac{1}{2}$.

20. 6th term in $(1-3x^2)^{-n}$ if $|x|<\frac{1}{2}$.

21. What is the first negative coefficient in the expansion of $(1+x)^{2\frac{1}{2}}$, where $|x|<1$?

22. How many terms are positive in the expansion of $(1-x^2)^{2\frac{1}{4}}$ where $|x|<1$?

23. How many terms are negative in the expansion of $(1-x^2)^{7\frac{1}{2}}$, where $|x|<1$?

Which is the numerically greatest term (or terms) in the expansions of the functions in Nos. 24-29? *Give the argument in full.*

24. $(1+x)^{10\frac{1}{2}}$ when $x=\frac{3}{4}$.
25. $(1-x)^{4\frac{2}{3}}$ when $x=0{\cdot}9$.
26. $(1+x)^{6\frac{1}{3}}$ when $x=\frac{2}{3}$.
27. $(1-x)^{-10}$ when $x=\frac{3}{4}$.
28. $(1+x)^{-\frac{1}{3}}$ when $x=\frac{1}{2}$.
29. $(1+x)^{-3\frac{1}{4}}$ when $x=0{\cdot}8$.

30. What can you say about the value of x if the 5th term of the expansion of $(1-x)^{-\frac{5}{2}}$ in ascending powers of x is the numerically greatest term?

31. Expand in ascending powers of x, as far as x^3,

$$(1-x)^{-1}-2(1-2x)^{-\frac{1}{2}}+(1-3x)^{-\frac{1}{3}}.$$

32. By expanding $(100-2)^{\frac{1}{2}}$, obtain the value of $\sqrt{2}$ to 5 places of decimals.

33. Expand in ascending powers of x as far as x^3, $(1-x+x^2)^{\frac{1}{2}}$.

34. If the expansion of $(1+x)^x$ can be deduced from that of $(1+x)^m$ by replacing m by x, verify that the first 4 terms are $1+x^2-\frac{1}{2}x^3+\frac{5}{6}x^4$.

35. Find the coefficient of x^3 in the expansion of

$$(1+3x)^{-\frac{1}{3}}(1+2x)^{\frac{1}{2}}.$$

36. If n is a positive integer, prove that the middle term of $(1+x)^{2n}$ is the same as the $(n+1)$th term of $(1-4x)^{-\frac{1}{2}}$.

37. Prove that the coefficient of x^r in $(1-x)^{-m-1}$ equals the coefficient of x^m in $(1-x)^{-r-1}$.

Forms of the Binomial Series

The examples in Exercise V.a show the various forms to which the binomial series may be reduced. It should be noted that coefficients which do not immediately appear to be of the form $\frac{m(m-1)(m-2)\dots}{1.2.3\dots}$ can often be manipulated so as to assume this form, as illustrated in Examples 4, 5, p. 87. It may assist the reader to see the reduction in the two following cases, $m=-n$ and $m=-\frac{p}{q}$.

If $|x|<1$,

$(1-x)^{-n}$ is the sum to infinity of

$$1+\frac{(-n)}{1}(-x)+\frac{(-n)(-n-1)}{1.2}(-x)^2 + \frac{(-n)(-n-1)(-n-2)}{1.2.3}(-x)^3+\dots$$

and we write

$$(1-x)^{-n}=1+\frac{n}{1}x+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\dots$$

Similarly, we have

$$(1+x)^{-n}=1-\frac{n}{1}x+\frac{n(n+1)}{1.2}x^2-\frac{n(n+1)(n+2)}{1.2.3}x^3+\dots,$$

and if we replace n by $\frac{p}{q}$, we have $(1-x)^{-\frac{p}{q}}$

$$=1+\frac{p}{q}x+\frac{p(p+q)}{q.2q}x^2+\frac{p(p+q)(p+2q)}{q.2q.3q}x^3+\dots$$

$$=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1.2}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{1.2.3}\left(\frac{x}{q}\right)^3+\dots.$$

The expansion of $(1+x)^{\frac{p}{q}}$ where $\frac{p}{q}$ is positive is sometimes less easy to recognise in numerical examples, because when the factors of each coefficient are all written positively, they decrease first and afterwards increase; this is illustrated by Example 2, p. 84.

Example 4. Find the sum to infinity of the series

$$1+\frac{10}{9}+\frac{10\,.\,16}{9\,.\,18}+\frac{10\,.\,16\,.\,22}{9\,.\,18\,.\,27}+\dots.$$

The factors in the numerators form an A.P. with common difference 6; we therefore divide each of these by 6.

The factors in the denominators form an A.P. with common difference 9; we therefore divide each by 9. Hence we have

$$1+\frac{\frac{5}{3}}{1}\left(\frac{6}{9}\right)+\frac{\frac{5}{3}\,.\,\frac{8}{3}}{1\,.\,2}\left(\frac{6}{9}\right)^2+\frac{\frac{5}{3}\,.\,\frac{8}{3}\,.\,\frac{11}{3}}{1\,.\,2\,.\,3}\left(\frac{6}{9}\right)^3+\dots;$$

and, since $\frac{6}{9}=\frac{2}{3}<1$, the sum to infinity of this series exists and equals $(1-\frac{2}{3})^{-\frac{5}{3}}$;

$$\therefore \text{ the sum to infinity}=(\tfrac{1}{3})^{-\frac{5}{3}}=3^{\frac{5}{3}}=3\sqrt[3]{9}.$$

Example 5. Find the sum to infinity of the series

$$\frac{1}{24}-\frac{1\,.\,3}{24\,.\,32}+\frac{1\,.\,3\,.\,5}{24\,.\,32\,.\,40}-\dots.$$

Proceeding as in Example 4 we write

$$s=\frac{\frac{1}{2}}{3}\cdot\left(\frac{2}{8}\right)-\frac{\frac{1}{2}\,.\,\frac{3}{2}}{3\,.\,4}\left(\frac{2}{8}\right)^2+\frac{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}}{3\,.\,4\,.\,5}\left(\frac{2}{8}\right)^3-\dots.$$

In order to express this in the standard binomial form, the factors 1 . 2 must be inserted in each denominator, and two additional factors must then be inserted in each numerator to secure that the number of factors in the numerator is the same as that in the denominator. In order that the factors of the numerator may remain in A.P., the additional factors (which should be the same in each term) must be $-\frac{3}{2}$, $-\frac{1}{2}$.

$$\therefore\ \frac{3s}{8}=\frac{\frac{-3}{2}\,.\,\frac{-1}{2}\,.\,\frac{1}{2}}{1\,.\,2\,.\,3}\,.\,\frac{1}{4}-\frac{\frac{-3}{2}\,.\,\frac{-1}{2}\,.\,\frac{1}{2}\,.\,\frac{3}{2}}{1\,.\,2\,.\,3\,.\,4}\,.\,\frac{1}{4^2}+\dots;$$

$$\therefore\ \frac{3s}{8}\,.\,\frac{1}{4^2}=-\frac{\frac{3}{2}\,.\,\frac{1}{2}\,.\,\frac{-1}{2}}{1\,.\,2\,.\,3}\,.\,\frac{1}{4^3}-\frac{\frac{3}{2}\,.\,\frac{1}{2}\,.\,\frac{-1}{2}\,.\,\frac{-3}{2}}{1\,.\,2\,.\,3\,.\,4}\,.\,\frac{1}{4^4}-\dots;$$

$$\therefore\ \frac{3s}{128} = -\left(1+\frac{1}{4}\right)^{\frac{3}{2}} + 1 + \frac{\frac{3}{2}}{1}\cdot\frac{1}{4} + \frac{\frac{3}{2}\cdot\frac{1}{2}}{1\,.\,2}\cdot\frac{1}{4^2}$$

$$= -\frac{5\sqrt{5}}{8} + 1 + \frac{3}{8} + \frac{3}{128} = -\frac{5\sqrt{5}}{8} + \frac{179}{128};$$

$$\therefore\ s = \tfrac{1}{3}(179 - 80\sqrt{5}).$$

EXERCISE V. b

What results are obtained by substitution and reduction in Nos. 1-4?

1. Put $x=\frac{1}{2}$ in $(1-x)^{-3}$ and in its expansion.

2. Put $x=\dfrac{1}{n}$ in $(1-x)^{-n}$ and in its expansion.

3. Put $x=\frac{1}{3}$ in $(1-x^2)^{-\frac{3}{2}}$ and in its expansion.

4. Put $x=\frac{3}{4}$ in $(1-x)^{\frac{1}{2}}$ and in its expansion.

Find the sum to infinity of the following series:

5. $1+6\left(\dfrac{4}{5}\right)+\dfrac{6\,.\,7}{1\,.\,2}\left(\dfrac{4}{5}\right)^2+\dfrac{6\,.\,7\,.\,8}{1\,.\,2\,.\,3}\left(\dfrac{4}{5}\right)^3+\dots.$

6. $1+2(\frac{1}{2})+3(\frac{1}{2})^2+4(\frac{1}{2})^3+\dots.$

7. $1+\frac{1}{2}(\frac{3}{4})+\dfrac{1\,.\,3}{2\,.\,4}\left(\dfrac{3}{4}\right)^2+\dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(\dfrac{3}{4}\right)^3+\dots.$

8. $1+\dfrac{n}{3}\cdot 2+\dfrac{n(n+1)}{3\,.\,6}\cdot 2^2+\dfrac{n(n+1)(n+2)}{3\,.\,6\,.\,9}\,.\,2^3+\dots.$

9. $1-\dfrac{1}{4}+\dfrac{1\,.\,3}{4\,.\,8}-\dfrac{1\,.\,3\,.\,5}{4\,.\,8\,.\,12}+\dots.$

10. $1-\dfrac{1}{5}+\dfrac{1\,.\,4}{5\,.\,10}-\dfrac{1\,.\,4\,.\,7}{5\,.\,10\,.\,15}+\dots.$

11. $1+\dfrac{1}{8}+\dfrac{1\,.\,3}{8\,.\,16}+\dfrac{1\,.\,3\,.\,5}{8\,.\,16\,.\,24}+\dots.$

12. $\dfrac{1}{2}\cdot\dfrac{1}{2^4}-\dfrac{1}{2\,.\,4}\cdot\dfrac{1}{2^8}+\dfrac{1\,.\,3}{2\,.\,4\,.\,6}\cdot\dfrac{1}{2^{12}}-\dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}\cdot\dfrac{1}{2^{16}}+\dots.$

13. $\dfrac{1}{3}+\dfrac{1\,.\,3}{3\,.\,6}+\dfrac{1\,.\,3\,.\,5}{3\,.\,6\,.\,9}+\dfrac{1\,.\,3\,.\,5\,.\,7}{3\,.\,6\,.\,9\,.\,12}+\dots.$

14. $\dfrac{3}{2}\left(\dfrac{1}{3}\right)+\dfrac{3\,.\,5}{2\,.\,3}\left(\dfrac{1}{3}\right)^2+\dfrac{3\,.\,5\,.\,7}{2\,.\,3\,.\,4}\left(\dfrac{1}{3}\right)^3+\dots.$

15. $\dfrac{4}{20}+\dfrac{4\,.\,7}{20\,.\,30}+\dfrac{4\,.\,7\,.\,10}{20\,.\,30\,.\,40}+\dots.$

16. $\dfrac{3}{2.4}+\dfrac{3.4}{2.4.6}+\dfrac{3.4.5}{2.4.6.8}+\dots$.

17. $\dfrac{1}{5}-\dfrac{1.4}{5.10}+\dfrac{1.4.7}{5.10.15}-\dots$.

18. $x^2+\dfrac{x^3}{3}+\dfrac{1.3}{3.4}x^4+\dfrac{1.3.5}{3.4.5}x^5+\dots,\ |x|<\tfrac{1}{2}$.

19. $1-n\left(\dfrac{1-y}{1+y}\right)+\dfrac{n(n+1)}{1.2}\left(\dfrac{1-y}{1+y}\right)^2$

$$-\frac{n(n+1)(n+2)}{1.2.3}\left(\frac{1-y}{1+y}\right)^3+\dots,\ y>0.$$

20. $1+n\left(1-\dfrac{1}{x}\right)+\dfrac{n(n+1)}{1.2}\left(1-\dfrac{1}{x}\right)^2$

$$+\frac{n(n+1)(n+2)}{1.2.3}\left(1-\frac{1}{x}\right)^3+\dots,\ x>\tfrac{1}{2}.$$

21. $1+\dfrac{1.3}{2.4}x^2+\dfrac{1.3.5.7}{2.4.6.8}x^4+\dots,\ |x|<1$.

22. Prove that the sum to infinity of

$$1+\frac{3}{8}+\frac{3.5}{8.10}+\frac{3.5.7}{8.10.12}+\dots$$

is double that of

$$\frac{1}{4}+\frac{1.3}{4.6}+\frac{1.3.5}{4.6.8}+\dots.$$

Partial Fractions

When the two fractions $\dfrac{3}{x+2}$ and $\dfrac{5}{x+1}$ are added together, the result is $\dfrac{8x+13}{(x+2)(x+1)}$; but this form is for many purposes less useful than the original sum. The opposite process of resolving a fraction like $\dfrac{8x+13}{(x+2)(x+1)}$ into the separate fractions $\dfrac{a}{x+2}+\dfrac{b}{x+1}$ is often required, for example when expanding in powers of x or in integration; it is called the expression of the function in **partial fractions.**

Example 6. Express $\dfrac{4x+7}{(x-2)(2x+1)}$ in partial fractions.

Assume that $\dfrac{4x+7}{(x-2)(2x+1)}\equiv\dfrac{a}{x-2}+\dfrac{b}{2x+1}$ where a, b are

constants (independent of x). This assumption is justified because it is equivalent to

$$4x+7 \equiv a(2x+1)+b(x-2), \quad \text{(i)}$$

or

$$4x+7 \equiv x(2a+b)+(a-2b),$$

and this will hold for all values of x if values of a, b exist such that $2a+b=4$ *and* $a-2b=7$. The solution of these equations gives $a=3$, $b=-2$, but usually it is better to find the values of the constants as follows :

Since (i) is an identity, it holds for any value of x we like to substitute.

If $x=2$, $8+7=5a$, $\therefore a=3$.

If $x=-\frac{1}{2}$, $-2+7=-2\frac{1}{2}b$, $\therefore 5b=-10$, $\therefore b=-2$.

$$\therefore \frac{4x+7}{(x-2)(2x+1)}=\frac{3}{x-2}-\frac{2}{2x+1}.$$

Example 7. Express $\dfrac{4x}{(x-1)(x+1)^2}$ in partial fractions.

It would be *wrong* to assume that

$$\frac{4x}{(x-1)(x+1)^2} \equiv \frac{a}{x-1}+\frac{b}{(x+1)^2}$$

where a, b are independent of x, because this is equivalent to

$$4x \equiv a(x+1)^2+b(x-1),$$

or

$$4x \equiv ax^2+x(2a+b)+a-b,$$

which will hold only if values of a, b exist such that $a=0$ *and* $2a+b=4$ *and* $a-b=0$. In general, *two* numbers cannot be found to satisfy *three* conditions, and here in particular it is obviously impossible.

It is, however, correct to assume that

$$\frac{4x}{(x-1)(x+1)^2} \equiv \frac{a}{x-1}+\frac{b}{(x+1)^2}+\frac{c}{x+1}$$

because this is equivalent to

$$4x \equiv a(x+1)^2+b(x-1)+c(x+1)(x-1), \quad \text{(i)}$$

or

$$4x \equiv x^2(a+c)+x(2a+b)+a-b-c,$$

which holds if values of a, b, c exist such that $a+c=0$ *and* $2a+b=4$ *and* $a-b-c=0$. In general, *three* numbers can be found to satisfy *three* conditions, and here in particular the solution of these

equations gives $a=1$, $b=2$, $c=-1$. But the values of a, b, c are best found as follows :

From the identity (i),

$$\text{if } x=1, \quad 4=4a, \quad \therefore\ a=1;$$
$$\text{if } x=-1, \quad -4=-2b, \quad \therefore\ b=2;$$

equate coefficients of x^2, $\therefore\ 0=a+c$, $\therefore\ c=-1$.

$$\therefore\ \frac{4x}{(x-1)(x+1)^2}=\frac{1}{x-1}+\frac{2}{(x+1)^2}-\frac{1}{x+1}.$$

It would also be correct to assume that

$$\frac{4x}{(x-1)(x+1)^2}\equiv\frac{\mathsf{A}}{x-1}+\frac{\mathsf{B}x+\mathsf{C}}{(x+1)^2}.$$

This form can be justified in the same way but is less convenient. It should also be noted that

$$\frac{\mathsf{B}x+\mathsf{C}}{(x+1)^2} \text{ can be written } \frac{\mathsf{B}(x+1)+\mathsf{C}-\mathsf{B}}{(x+1)^2}$$

which reduces to
$$\frac{\mathsf{B}}{x+1}+\frac{\mathsf{C}-\mathsf{B}}{(x+1)^2}.$$

Example 8. Express $\dfrac{5x-7}{(x+3)(x^2+2)}$ in partial fractions.

It would be *wrong* to assume that

$$\frac{5x-7}{(x+3)(x^2+2)}\equiv\frac{a}{x+3}+\frac{b}{x^2+2}$$

where a, b are independent of x, because as in Example 7 this requires that the *two* unknowns a, b must satisfy *three* conditions. But the arguments used above show that it is correct to assume

$$\frac{5x-7}{(x+3)(x^2+2)}\equiv\frac{a}{x+3}+\frac{bx+c}{x^2+2}.$$

which is equivalent to

$$5x-7\equiv a(x^2+2)+(bx+c)(x+3).$$

If $x=-3$, $-15-7=11a$, $\therefore\ a=-2$.

If $x=0$, $-7=2a+3c$, $\therefore\ 3c=-7+4$, $\therefore\ c=-1$.

Equate coefficients of x^2, $\therefore\ 0=a+b$, $\therefore\ b=2$.

$$\therefore\ \frac{5x-7}{(x+3)(x^2+2)}=-\frac{2}{x+3}+\frac{2x-1}{x^2+2}.$$

Alternatively, after finding the value of a, we may continue

$$5x - 7 + 2(x^2 + 2) \equiv (bx + c)(x + 3) ;$$

$$\therefore\ 2x^2 + 5x - 3 \equiv (x + 3)(2x - 1) \equiv (bx + c)(x + 3) ;$$

$$\therefore\ bx + c \equiv 2x - 1.$$

Example 9. Decompose $\dfrac{x^3 + x^2 + 6}{x^2 - 1}$.

Since the degree of the numerator is not less than that of the denominator, we must start by dividing, as with improper fractions in arithmetic. We then find

$$\frac{x^3 + x^2 + 6}{x^2 - 1} = x + 1 + \frac{x + 7}{x^2 - 1},$$

and then by the method of Example 6 we obtain

$$\frac{x^3 + x^2 + 6}{x^2 - 1} = x + 1 + \frac{4}{x - 1} - \frac{3}{x + 1}.$$

Alternatively, it is sometimes convenient to combine the division with the rest of the work by assuming, for example,

$$\frac{x^3 + x^2 + 6}{x^2 - 1} = ax + b + \frac{c}{x - 1} + \frac{d}{x + 1}.$$

Example 10. Express $\dfrac{x^2 - 8x + 9}{(x + 1)(x - 2)^3}$ in partial fractions.

First Method.

Assume $\dfrac{x^2 - 8x + 9}{(x + 1)(x - 2)^3} \equiv \dfrac{a}{x + 1} + \dfrac{b}{(x - 2)^3} + \dfrac{c}{(x - 2)^2} + \dfrac{d}{x - 2}$,

then $x^2 - 8x + 9 \equiv a(x - 2)^3 + b(x + 1) + c(x + 1)(x - 2)$
$+ d(x + 1)(x - 2)^2$.

If $x = -1$, $1 + 8 + 9 = -27a$, $\therefore\ a = -\frac{2}{3}$;

if $x = 2$, $4 - 16 + 9 = 3b$, $\therefore\ b = -1$;

equate coefficients of x^3, $\therefore\ 0 = a + d$, $\therefore\ d = \frac{2}{3}$;

if $x = 0$, $9 = -8a + b - 2c + 4d$, $\therefore\ 2c = -9 + \frac{16}{3} - 1 + \frac{8}{3}$;

$\therefore\ c = -1.$

$$\therefore\ \text{expression} = -\frac{2}{3(x + 1)} - \frac{1}{(x - 2)^3} - \frac{1}{(x - 2)^2} + \frac{2}{3(x - 2)}.$$

Second Method.

If, as here, the denominator contains a factor repeated three or more times, it may be shorter to proceed as follows :

Put $x-2=y$, then the expression becomes

$$\frac{(y+2)^2-8(y+2)+9}{y^3(y+3)}=\frac{y^2-4y-3}{y^3(y+3)}.$$

But $-3-4y+y^2=-(3+y)-3y+y^2=-(3+y)-y(3+y)+2y^2$

$$=-(3+y)-y(3+y)+\tfrac{2}{3}y^2(3+y)-\frac{2y^3}{3};$$

$$\therefore\ \frac{y^2-4y-3}{y^3(y+3)}=-\frac{1}{y^3}-\frac{1}{y^2}+\frac{2}{3y}-\frac{2}{3(y+3)},$$

which gives the same result as before.

Note. The process of expressing functions in partial fractions, although elementary, is often laborious. In such cases, the alternative methods given in Examples 8, 10 are preferable.

EXERCISE V. c

Express in partial fractions :

1. $\dfrac{5x+5}{(x-1)(x+4)}$.

2. $\dfrac{2+x}{x-x^2}$.

3. $\dfrac{7x-8}{(x+1)(4x-1)}$.

4. $\dfrac{3x+1}{(x+2)^2}$.

5. $\dfrac{x+10}{x^2-x-12}$.

6. $\dfrac{x-1}{(x+1)^2}$.

7. $\dfrac{x^2+1}{x^2-1}$.

8. $\dfrac{1}{x^2(x+3)}$.

9. $\dfrac{x^2-4x+2}{x^2-5x+6}$.

10. $\dfrac{3x^2-2x-3}{(x-1)(x-2)(x-3)}$.

11. $\dfrac{6x^2}{(x^2-1)(x-2)}$.

12. $\dfrac{9x-8}{x(x-2)^2}$.

13. $\dfrac{6x-3}{(x+1)^2(x-2)}$.

14. $\dfrac{3x+1}{(x-1)(x^2-1)}$.

15. $\dfrac{5x^2-11x+8}{(1-x)(2-x)^2}$.

16. $\dfrac{x^2-1}{(x-3)^3}$.

17. $\dfrac{x+2}{x(x^2+1)}$.

18. $\dfrac{x^4+x^2}{x^2-1}$.

19. $\dfrac{7x^2+12x+2}{(x+1)^3(x-2)}$.

20. $\dfrac{2x^2+3x+1}{(x-2)^3(x-3)}$.

21. $\dfrac{3x+3}{(x-1)(x^2+x+1)}$.

22. $\dfrac{12(x^2+x-3)}{(x^2-1)(x^2-4)}$.

23. $\dfrac{1-2x}{x^3+1}$.

24. $\dfrac{4x^2}{(x-1)^3(x^2+1)}$.

25. $\dfrac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}$.

26. $\dfrac{a-b}{(x-a)^2(x-b)}$.

27. Use the method of partial fractions to prove that

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)}=\frac{1}{x}-\frac{{}_nC_1}{x+1}+\frac{{}_nC_2}{x+2}-\ldots+(-1)^n\frac{{}_nC_n}{x+n}.$$

What special results are obtained by putting (i) $x=1$, (ii) $x=2$, (iii) $x=\frac{1}{2}$.

28. Prove that

$$\frac{x^3}{(x-a)(x-b)}-\frac{a^3}{(a-b)(x-a)}-\frac{b^3}{(b-a)(x-b)}=x+a+b.$$

29. Express $\dfrac{x^3+4x-3}{x^4+x^2+1}$ in partial fractions.

30. Express $\dfrac{n+10}{n(n+3)(n+5)}$ in partial fractions, and deduce the sum to infinity of the series whose nth term is $\dfrac{n+10}{n(n+3)(n+5)}$.

Example 11. Find the coefficient of x^r in the expansion in ascending powers of x of $\dfrac{3-x}{(1-2x)^2}$.

If $|x|<\frac{1}{2}$, $(1-2x)^{-2}=1+2(2x)+3(2x)^2+\ldots+(r+1)(2x)^r+\ldots$;

$\therefore$ the coefficient of x^r in $(3-x)(1-2x)^{-2}$ is

$$3(r+1).2^r-r.2^{r-1}=2^{r-1}(6r+6-r)=2^{r-1}(5r+6).$$

Example 12. Find the coefficient of x^r in the expansion in ascending powers of x of $\dfrac{1}{(1-2x)(3+x)}$.

$$\frac{1}{(1-2x)(3+x)}=\frac{2}{7(1-2x)}+\frac{1}{7(3+x)}$$

$$=\tfrac{2}{7}(1-2x)^{-1}+\frac{1}{21}\left(1+\frac{x}{3}\right)^{-1};$$

$\therefore$ if $|x|<\frac{1}{2}$, the expression

$$=\tfrac{2}{7}(1+2x+2^2x^2+\ldots)+\frac{1}{21}\left(1-\frac{x}{3}+\frac{x^2}{3^2}-\ldots\right).$$

$\therefore$ the coefficient of $x^r=\frac{1}{7}.2^{r+1}+(-1)^r.\dfrac{1}{21}.\dfrac{1}{3^r}$

$$=\tfrac{1}{7}\{2^{r+1}+(-1)^r.3^{-r-1}\}.$$

EXERCISE V. d

State the condition for expanding the following functions in ascending powers of x, and find the coefficient of x^r.

1. $\dfrac{x}{(2-x)^2}$. 2. $\dfrac{1-2x}{(1-x)^2}$. 3. $\dfrac{1+x}{(1-x)^3}$.

4. $\dfrac{x}{(1-2x)(1-3x)}$. 5. $\dfrac{1}{(3-x)(4-x)}$.

6. $\dfrac{9}{(1+2x)(4-x)}$.

7. $\dfrac{(1+x)^2}{(1-x)^4}$.

8. $\dfrac{1+x}{1+x-2x^2}$.

9. $\dfrac{2-3x}{1-3x+2x^2}$.

10. $\dfrac{9x}{(x+2)(2x+1)^2}$.

11. $\dfrac{12x-34}{(x-2)(x+3)(x-4)}$.

12. $\dfrac{a-b}{(1-ax)(1-bx)}$.

13. $\dfrac{(a-b)(b-c)(c-a)}{(1-ax)(1-bx)(1-cx)}$.

14. If $|x|<1$, expand in powers of x, $\dfrac{1+x}{(1-x)^3}+\dfrac{1-x}{(1+x)^3}$.

What result is obtained by taking $x=\frac{1}{2}$?

15. Prove that the coefficient of x^r in the expansion of $\left(\dfrac{1+x}{1-x}\right)^2$ is $4r^2+2$.

16. If $\dfrac{5x-3}{(x-1)^2(x+1)}$ is expanded in powers of x, prove that the coefficient of x^r is $r-3$ or $r+1$, according as r is even or odd.

17. What is the coefficient of x^r in the expansion of

$$(1+x+x^2)^{-1} \text{ ? } \left[\text{Note that } 1+x+x^2=\frac{1-x^3}{1-x}.\right]$$

18. What is the coefficient of x^r in the expansion of $\dfrac{1-x}{1-x+x^2}$?

19. What are the conditions which allow $\dfrac{1}{x^2-5x+6}$ to be expanded in the forms

$$\text{(i) } p_0+p_1x+p_2x^2+\dots\text{ ; } \quad \text{(ii) } q_0+\frac{q_1}{x}+\frac{q_2}{x^2}+\dots\text{ ?}$$

Prove that $q_{n+1}=6^n p_{n-1}$.

20. If n is a positive integer, find the coefficient of x^n in the expansion of $\dfrac{(1+2x)^n}{1-x}$.

[Replace $1+2x$ by $3-2(1-x)$.]

21. If n is a positive integer, prove that the coefficient of x^n in the expansion of $\dfrac{(3x-2)^n}{(1-x)^2}$ is $1-2n$. [Use the method of No. 20.)

22. What is the coefficient of x^r in the expansion of $\dfrac{\sqrt{(1+x)}}{\sqrt{(1-x)}}$ in ascending powers of x ?

23. If $\dfrac{(a-b)^2}{(1-ax)^2(1-bx)}$ can be expanded in ascending powers of x, prove that the coefficient of x^r is

$$(r+1)a^{r+2}-(r+2)a^{r+1}b+b^{r+2}.$$

Approximations

The practical utility of an infinite series depends on the rapidity with which successive terms diminish. The most useful series are those in which the sum of the first few terms is a good approximation for its sum to infinity; such series are said to converge rapidly.

Example 13. Evaluate $\sqrt{2}$ to 6 places of decimals.

Since $100-2=98=7^2 . 2$, $7\sqrt{2}=(100-2)^{\frac{1}{2}}=10(1-0{\cdot}02)^{\frac{1}{2}}$;

$$\therefore\ \frac{7\sqrt{2}}{10}=1-\tfrac{1}{2}(0{\cdot}02)-\tfrac{1}{8}(0{\cdot}02)^2-\tfrac{1}{16}(0{\cdot}02)^3-\tfrac{5}{128}(0{\cdot}02)^4-\dots$$
$$=1-0{\cdot}01-0{\cdot}000{,}05-0{\cdot}000{,}000{,}5-0{\cdot}000{,}000{,}006\dots$$
$$=0{\cdot}989949494\dots;$$
$$\therefore\ \sqrt{2}=1{\cdot}41421356\dots.$$
$$\therefore\ \sqrt{2}\simeq 1{\cdot}414214 \text{ correct to 6 places of decimals.}$$

Degree of Accuracy

Since the successive binomial coefficients steadily decrease *after a certain stage*, we can estimate the degree of error caused by considering only the first few terms. Thus, if in Example 13, we had taken only 4 terms instead of 5 terms, the error in the value of $\frac{7\sqrt{2}}{10}$ would certainly be less than

$$\tfrac{5}{128}[(0{\cdot}02)^4+(0{\cdot}02)^5+(0{\cdot}02)^6+\dots]$$

and this is less than

$$\frac{5(0{\cdot}02)^4}{128(1-0{\cdot}02)}<0{\cdot}000{,}000{,}007;$$

and therefore the error in the value of $\sqrt{2}$ would be less than $0{\cdot}000{,}000{,}01$, which shows that, for the required approximation 4 terms are sufficient.

Example 14. Find an approximation for $\frac{\sqrt{(1+2x)}}{\sqrt[3]{(1+3x)}}$, if x is so small that terms involving x^3 may be neglected.

$$(1+2x)^{\frac{1}{2}}\simeq 1+x-\tfrac{1}{8}(2x)^2=1+x-\tfrac{1}{2}x^2,$$
$$(1+3x)^{-\frac{1}{3}}\simeq 1-x+\tfrac{2}{9}(3x)^2=1-x+2x^2;$$
$$\therefore\ \frac{\sqrt{(1+2x)}}{\sqrt[3]{(1+3x)}}\simeq(1+x-\tfrac{1}{2}x^2)(1-x+2x^2)$$
$$\simeq 1+\tfrac{1}{2}x^2.$$

Note. If x^3 and higher powers of x are neglected, the error is said to be "*of order* x^3."

Example 15. If x is large, find an approximation for

$$\sqrt{(x^2+1)} - \sqrt{(x^2-1)}.$$

$$(x^2+1)^{\frac{1}{2}} = x\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}$$

$$\simeq x\left(1+\frac{1}{2x^2}-\frac{1}{8x^4}\right), \text{ neglecting } \frac{1}{x^6}, \text{ etc.}$$

Similarly, $(x^2-1)^{\frac{1}{2}} \simeq x\left(1-\frac{1}{2x^2}-\frac{1}{8x^4}\right)$;

$$\therefore\ \sqrt{(x^2+1)} - \sqrt{(x^2-1)} \simeq \frac{1}{x}, \text{ neglecting } \frac{1}{x^5}, \text{ etc.}$$

The error involved in this approximation is therefore of order $\frac{1}{x^5}$. If a more precise estimate of the degree of accuracy is required, we may say that the error in the expression taken for $(x^2+1)^{\frac{1}{2}}$ is less than

$$x\left[\frac{1}{16x^6}+\frac{1}{16x^8}+\frac{1}{16x^{10}}+\dots\right] < \frac{1}{16x^5}\cdot\frac{1}{1-\frac{1}{x^2}},$$

and therefore the total error $< \dfrac{1}{8x^3(x^2-1)}$.

Example 16. If $p-q$ is small compared with q, obtain an approximation for $\sqrt[n]{\frac{p}{q}}$.

$$\frac{p}{q} = \left(1+\frac{p-q}{p+q}\right) \div \left(1-\frac{p-q}{p+q}\right);$$

$$\therefore\ \sqrt[n]{\frac{p}{q}} = \left(1+\frac{p-q}{p+q}\right)^{\frac{1}{n}} \div \left(1-\frac{p-q}{p+q}\right)^{\frac{1}{n}}$$

$$\simeq \left\{1+\frac{p-q}{n(p+q)}\right\} \div \left\{1-\frac{p-q}{n(p+q)}\right\};$$

$$\therefore\ \sqrt[n]{\frac{p}{q}} \simeq \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}.$$

If we put $p-q=qx$, so that x is small, this result becomes

$$\sqrt[n]{(1+x)} \simeq \frac{2n+(n+1)x}{2n+(n-1)x}.$$

It can be shown (Ex. V. c, No. 15) that the error is of order x^3.

EXERCISE V. e

1. Evaluate $\sqrt{1 \cdot 02}$ to 5 places of decimals.

2. Evaluate $\sqrt[3]{999}$ to 5 places of decimals.

3. Evaluate $\sqrt[5]{3250}$ to 5 places of decimals.

4. Use the expansion of $(1+\frac{3}{125})^{\frac{1}{3}}$ to evaluate $\sqrt[3]{2}$ to 5 places of decimals.

5. The surface S sq. in. of a sphere is connected with the volume V cu. in. by the relation, $S^3 = 36\pi V^2$. If the volume of a spherical soap bubble is increased by 10 per cent., show that the surface is increased by about 6·56 per cent.

Prove the following approximations if x is small:

6. $(1-x)^{-\frac{1}{2}} - (1+x)^{\frac{1}{2}} \simeq \frac{1}{2}x^2 + \frac{1}{4}x^3.$

7 $\sqrt{\left(\dfrac{1-x}{1+x}\right)} \simeq 1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3.$

8. $\sqrt{(1-3x+2x^2)} \simeq 1 - \dfrac{3x}{2} - \dfrac{x^2}{8} - \dfrac{3x^3}{16}.$

9. $\sqrt[3]{\left(\dfrac{1+x}{1-x}\right)} \simeq 1 + \frac{2}{3}x + \frac{2}{9}x^2 + \frac{22}{81}x^3.$

10. $\sqrt[3]{(1+3x)} \div \sqrt{(1-2x)} \simeq 1 + 2x + \frac{3}{2}x^2 + \frac{14}{3}x^3.$

11. $(1+ax)^p(1+bx)^q \div (1+cx)^r \simeq 1 + (ap+bq-cr)x.$

12. $\sqrt[3]{(1+x)} - \dfrac{3+2x}{3+x} \simeq \dfrac{2x^3}{81}.$

13. $\{(1-x)^{-\frac{1}{2}} - (1+x)^{\frac{1}{2}}\} \div \{(1-\frac{1}{2}x)^{-\frac{1}{2}} - (1+\frac{1}{2}x)^{\frac{1}{2}}\} \simeq 4 + x.$

14. $\{(1+x)^2 - 2(1+x)^6 + (1+x)^{10}\}$
$\div \{(1+2x) - 2(1+2x)^3 + (1+2x)^5\} \simeq 1.$

15. $\sqrt[n]{(1+x)} - \dfrac{2n+(n+1)x}{2n+(n-1)x} \simeq \dfrac{n^2-1}{12n^3}x^3.$

16. If x is large, prove that

$$\sqrt[3]{(x^3+6)} - \sqrt[3]{(x^3+3)} \simeq \frac{1}{x^2} - \frac{3}{x^5}.$$

17. Find a rough approximation to the small positive value of x which satisfies $\dfrac{(1+x)^7}{(1-x)^4} = 1 \cdot 05.$

18. If x is so large that $\dfrac{1}{x^4}$ is negligible, obtain an approximation for $\dfrac{x\sqrt{(x^2-2x)}}{(x+1)^2}.$

19. If $(p-q)^4 \div q^4$ is negligible, show that

$$\sqrt{\frac{p}{q}} = \frac{p}{p+q} + \frac{p+q}{4q}.$$

20. If x is nearly equal to 1, prove that

$$\frac{\sqrt{x} - \sqrt[3]{x}}{\sqrt[3]{x} - \sqrt[4]{x}} \simeq 4.$$

21. If $t\sqrt{\frac{g}{l}} = k = \mathsf{T}\sqrt{\frac{\mathsf{G}}{\mathsf{L}}}$, and if the values of L, G differ but slightly from those of l, g respectively, prove that

$$\frac{\mathsf{T}-t}{\mathsf{T}} \simeq \frac{1}{2}\left(\frac{\mathsf{L}-l}{\mathsf{L}} - \frac{\mathsf{G}-g}{\mathsf{G}}\right).$$

22. If ac is small compared with b^2, show that

$$-\frac{c}{b} - \frac{ac^2}{b^3} \quad \text{and} \quad -\frac{b}{a} + \frac{c}{b} + \frac{ac^2}{b^3}$$

are approximate values of the roots of $ax^2 + bx + c = 0$.

Homogeneous Products

The various products of degree r which can be formed with any or all of n given letters are called the *homogeneous products of r dimensions formed from n letters*; it is to be understood that repetitions of a letter are allowed. Thus the homogeneous products of 2 dimensions formed from the 3 letters a, b, c, are

$$a^2,\ b^2,\ c^2,\ bc,\ ca,\ ab.$$

These are the terms in the coefficient of x^2 in the expansion of $(1 + ax + a^2x^2)(1 + bx + b^2x^2)(1 + cx + c^2x^2)$.

Similarly, the homogeneous products of r dimensions formed from the 3 letters a, b, c are the terms in the coefficient of x^r in the expansion of

$$(1 + ax + a^2x^2 + \dots + a^m x^m)(1 + bx + \dots + b^m x^m)(1 + cx + \dots + c^m x^m)$$

if $m \geqslant r$.

This expression

$$= \frac{(1 - a^{m+1}x^{m+1})(1 - b^{m+1}x^{m+1})(1 - c^{m+1}x^{m+1})}{(1-ax)(1-bx)(1-cx)}.$$

$\therefore$ the *sum* of all the homogeneous products is the coefficient of x^r in this expression and, since $m + 1 > r$, this is the same as the coefficient of x^r in the expansion of $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$,

which may be found (see Ex. V. f, No. 30) by the method of partial fractions.

If we put $a = b = c = 1$, each term in the coefficient of x^r becomes 1, and therefore the *number of homogeneous products of r dimensions* formed from 3 letters is the coefficient of x^r in the expansion of $\dfrac{1}{(1-x)^3} = (1-x)^{-3}$, namely $\frac{1}{2}(r+1)(r+2)$.

This method can be applied to any number of letters.

The homogeneous products of r dimensions formed from n letters are the terms in the coefficient of x^r in the expansion of the n factors

$$(1 + a_1x + a_1{}^2x^2 + \dots + a_1{}^mx^m)\dots(1 + a_nx + a_n{}^2x^2 + \dots + a_n{}^mx^m),$$

where $m \geq r$, and this is the same as the coefficient of x^r in the expansion of

$$\frac{1}{(1-a_1x)(1-a_2x)\dots(1-a_nx)}.$$

$\therefore$ the *number* of homogeneous products of r dimensions formed from n letters is the coefficient of x^r in the expansion of $\dfrac{1}{(1-x)^n} = (1-x)^{-n}$; this number is denoted by ${}_n\mathrm{H}_r$. We therefore have

$$\mathbf{{}_nH_r = \frac{n(n+1)(n+2)\dots(n+r-1)}{1 \,.\, 2 \,.\, 3 \dots r} = \frac{(n+r-1)!}{r!(n-1)!}}.$$

This result shows that

$${}_n\mathrm{H}_r = {}_{n+r-1}\mathrm{C}_r$$

and can be established independently by the principles of Ch. I. Consider the following example:

Example 17. Find the number of ways in which a row of 13 dots can be divided into groups by 9 strokes, if the strokes need not be separated by dots and may come at either end.

One such method of partition is as follows:

| . . | . | . . . | | . | | | . . | . . . .

The total number of ways is the same as the number of ways of arranging, say, 9 A's and 13 B's in a row, and, by p. 7, this is

$$\frac{22!}{9!\,13!} = {}_{22}\mathrm{C}_{13}.$$

Suppose the dots all written down first and that the strokes are then inserted in succession from the left. In the particular partition shown above, the strokes cut off in succession

$$0,\ 2,\ 1,\ 3,\ 0,\ 1,\ 0,\ 0,\ 2 \text{ dots},$$

leaving 4 dots at the end.

This corresponds to the term

$$a^0b^2c^1d^3e^0f^1g^0h^0i^2j^4,$$

which is a product of degree 13 formed from the 10 letters $a, b, c, \ldots i, j$. Each homogeneous product of degree 13 formed from these 10 letters corresponds to one mode of partition, and conversely. Therefore the number of these homogeneous products ${}_{10}\mathrm{H}_{13}$ equals the number of modes of partition, ${}_{22}\mathrm{C}_{13}$.

The same argument may be used for any number of dots and strokes.

The number of ways in which a row of r dots can be divided into groups by $n-1$ strokes is ${}_{r+n-1}\mathrm{C}_r$, and this corresponds to the number of homogeneous products of degree r that can be formed from n letters, namely ${}_n\mathrm{H}_r$.

Example 18. In how many ways can a total of 15 be thrown with four dice of different colours, whose faces are numbered 1, 2, 3, 4, 5, 6 ?

Consider the expression

$$(x+x^2+x^3+\ldots+x^6)(x+x^2+\ldots+x^6)(x+x^2+\ldots+x^6) \\ \times(x+x^2+\ldots+x^6).$$

Any term $x^p \cdot x^q \cdot x^r \cdot x^s$ in the product equals x^{15} if

$$p+q+r+s=15;$$

therefore the number of these terms is the same as the number of solutions of the equation $p+q+r+s=15$, where p, q, r, s may have any of the values 1, 2, 3, 4, 5, 6.

$\therefore$ the required number is the coefficient of x^{15} in

$$(x+x^2+x^3+\ldots+x^6)^4,$$

which equals $\left\{\dfrac{x(1-x^6)}{1-x}\right\}^4$.

This equals the coefficient of x^{11} in $(1-4x^6+6x^{12}-\ldots)(1-x)^{-4}$;

$$\therefore \text{ the number} = \frac{12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3} - 4 \cdot \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3} = 140.$$

Sum of the First r Coefficients of a Power Series

Suppose that $\quad a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots$

is absolutely convergent for $|x| < k$ with sum to infinity $f(x)$.

Then $a_0 + a_1 + a_2 + \ldots + a_{r-1}$ is the coefficient of x^{r-1} when the product $\quad (1 + x + x^2 + \ldots + x^{r-1})(a_0 + a_1x + a_2x^2 + \ldots)$

is rearranged as a power series in x. It can be proved that this process is legitimate if the series $\Sigma |a_rx^r|$ is convergent, and the sum to infinity is then $\dfrac{(1-x^r)f(x)}{1-x}$. But the coefficient of x^{r-1} in the expansion of this function is the same as that of x^{r-1} in the expansion of $\dfrac{f(x)}{1-x}$.

$\therefore$ $a_0 + a_1 + \ldots + a_{r-1}$ equals the coefficient of x^{r-1} in the expansion of $\dfrac{f(x)}{1-x}$.

Example 19. Given

$$(1-x)^m = 1 - m_1x + m_2x^2 - m_3x^3 + \ldots, \ |x| < 1,$$

find the value of $1 - m_1 + m_2 - m_3 + \ldots + (-1)^r m_r$.

The expression equals the coefficient of x^r in the expansion of

$$\frac{(1-x)^m}{1-x} = (1-x)^{m-1}.$$

It therefore equals $(-1)^r . (m-1)_r$.

EXERCISE V. f

Describe, as on p. 99, the nature of the following coefficients in Nos. 1-3.

1. Coefficient of x^3 in

$$(1 + \sum_1^3 a^r x^r)(1 + \sum_1^3 b^r x^r)(1 + \sum_1^3 c^r x^r)(1 + \sum_1^3 d^r x^r).$$

2. Coefficient of x^{12} in

$$(ax + a^2x^2 + \ldots)(bx + b^2x^2 + \ldots)(cx + c^2x^2 + \ldots).$$

3. Coefficient of x^6 in $\sum_2^8 (a^r x^r) . \sum_1^8 (b^r x^r) . \sum_0^8 (c^r x^r)$.

4. Find the coefficient of (i) x^4; (ii) x^7 in the expansion of $(1 + x + x^2 + x^3 + x^4)^3$.

5. Find the coefficient of (i) x^5; (ii) x^{10} in the expansion of $(x+x^2+x^3+x^4)^3$.

6. Find the number of homogeneous products of 4 dimensions formed from 10 letters.

7. How many terms are there in the expansion of

$$\text{(i) } (a_1+a_2+a_3+a_4)^5; \quad \text{(ii) } (a_1+a_2+\ldots+a_n)^k$$

where k is a positive integer.

8. How many unequal triangles can be drawn if every side is either 5, 6, 7 or 8 cm. long?

9. A man receives from a photographer 4 different proofs of his photograph. In how many ways can he select 6 copies?

10. In how many ways can a total of 10 be thrown with three dice of different colours, the faces being numbered 1, 2, 3, 4, 5, 6?

11. In how many ways can 5 like things be arranged in a row of 3 pigeon-holes (i) if there is no restriction, (ii) if there is at least one thing in each pigeon-hole?

12. How many numbers less than a million have 6 as the sum of their digits?

13. How many solutions are there of the equation

$$a+b+c+d+e=12,$$

if each of the letters must have one of the values 1, 2, 3, 4?

14. How many selections of 8 letters can be made from 5 a's, 6 b's, 8 c's, if each selection contains at least 1 a, 1 b, 2 c's?

15. In an examination there are 3 papers for each of which the maximum mark is 100. Find the number of ways in which a candidate can obtain a total of 200 marks for the 3 papers.

16. By writing $1+x^2$ in the form $2x+(1-x)^2$, find the coefficient of x^{2n} in $\dfrac{(1+x^2)^n}{(1-x)^4}$.

17. If n and r are positive integers, prove that the coefficient of x^{n+r} in $\dfrac{(1-3x)^n}{1-4x}$ is 4^r.

18. Prove that ${}_n\mathrm{H}_r={}_{n-1}\mathrm{H}_r+{}_n\mathrm{H}_{r-1}$.

19. What is the coefficient of x^r in $(1+x+x^2+\ldots+x^{m-1})^n$ if $r<m$? Find also the coefficient of x^{m+2}.

20. Express in two ways the coefficient of x^{r-1} in the expansion of $(1-x)^{-2}(1-x)^{-3}$.

21. Use the product $(1-x)^{-1}.(1-x)^{-\frac{1}{2}}$ to find the value of

$$1+\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\ldots+\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}.$$

22. Prove that, if $|x| < 1$,

$$\frac{1+x}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots,$$

and deduce that, if n is a positive integer greater than 2,

$$1 - {}_nC_1 \,.\, 2^2 + {}_nC_2 \,.\, 3^2 - {}_nC_3 \,.\, 4^2 + \dots = 0.$$

23. Use the product $(1-x)^n \,.\, (1-x)^{-2}$ to find the value of

$$m - (m-1)n + (m-2)\cdot\frac{n(n-1)}{1\,.\,2} - (m-3)\cdot\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} + \dots,$$

to m terms, where m, n are positive integers, $m < n$.

24. If $a_r = \dfrac{1\,.\,3\,.\,5 \dots (2r-1)}{2\,.\,4\,.\,6 \dots 2r}$, and if n is odd, prove that

$$a_n + a_1a_{n-1} + a_2a_{n-2} + \dots, \text{ to } \tfrac{1}{2}(n+1) \text{ terms}, = \tfrac{1}{2}.$$

25. Find the sum of the first r coefficients in the expansion of $\dfrac{2x+4}{(1+x)(1-2x)}$ in ascending powers of x.

26. Use the product $\left(1 - \dfrac{1}{x}\right)^m (1-x)^{-m}$ where m is a positive integer to prove that

$$1 - \frac{m^2}{1^2} + \frac{m^2(m^2-1^2)}{1^2\,.\,2^2} - \frac{m^2(m^2-1^2)(m^2-2^2)}{1^2\,.\,2^2\,.\,3^2} + \dots$$

to $(m+1)$ terms is zero.

27. If m, n are positive integers, $m < n-1$, find the value of

$$(m+1)m - {}_nC_1 m(m-1) + {}_nC_2(m-1)(m-2) - \dots$$

to m terms.

28. If n is a positive integer, prove that

$$n + (n-1)\tfrac{1}{3} + (n-2)\frac{1\,.\,4}{3\,.\,6} + (n-3)\frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9} + \dots$$

to n terms equals $\dfrac{7\,.\,10\,.\,13 \dots (3n+1)}{3^{n-1}\,.\,(n-1)!}$.

29. By expanding $(1-2x+x^2)^{-1}$ in two different ways, prove that if n is a positive integer

$$2^n - (n-1)2^{n-2} + \frac{(n-2)(n-3)}{1\,.\,2}2^{n-4} - \frac{(n-3)(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-6} + \dots$$

equals $n+1$.

30. Prove that the sum of the homogeneous products of r dimensions that can be formed from the 3 letters a, b, c is

$$-\{a^{r+2}(b-c) + b^{r+2}(c-a) + c^{r+2}(a-b)\} \div \{(a-b)(b-c)(c-a)\}.$$

CHAPTER VI

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Functions defined by Definite Integrals

The indefinite integral, $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ for all values of n, *except* $n = -1$.

It is convenient to consider the definite integral

$$\int_1^t x^n\,dx, \text{ where } t > 0.$$

If $n \neq -1$, this equals $\left[\dfrac{x^{n+1}}{n+1}\right]_1^t = \dfrac{t^{n+1}-1}{n+1}$.

If $n = -1$, the integral becomes $\int_1^t \dfrac{1}{x}\,dx$, and this cannot be expressed in terms of elementary functions. It represents the area between the graph of $y = \dfrac{1}{x}$, the x axis, and the ordinates $x = 1$, $x = t$, where we suppose $t > 0$.

Approximate values, for given values of t, can be found by the ordinary methods of practical geometry such as counting squares, Simpson's rule, etc., but more effective methods will be given later in this chapter.

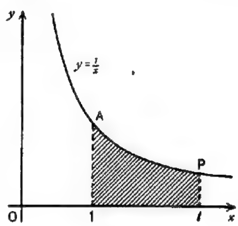

FIG. 2.

Since this curve is a rectangular hyperbola, we shall for the present denote this area function by the symbol hyp(t), and write

$$\text{hyp}(t) \equiv \int_1^t \frac{1}{x}\,dx, \quad t > 0, \quad \text{.......................}(1)$$

The definition of hyp(t) requires that $t > 0$. If $t \leqslant 0$, we shall not discuss or even define hyp(t), and whenever this function is used, it is to be understood that $t > 0$.

From the definition, it follows that $\operatorname{hyp}(1)=0$ and that $\operatorname{hyp}(t)$ steadily increases as t increases.

Further, if x lies between 1 and $1+u$ where $1+u>0$, $\dfrac{1}{x}$ lies between 1 and $\dfrac{1}{1+u}$.

$\therefore$ $\operatorname{hyp}(1+u)\equiv\displaystyle\int_1^{1+u}\frac{1}{x}dx$ lies between

$$\int_1^{1+u}dx \text{ and } \frac{1}{1+u}\int_1^{1+u}dx;$$

that is, between u and $\dfrac{u}{1+u}$;

$$\therefore\ \frac{u}{1+u}<\operatorname{hyp}(1+u)<u, \qquad\text{.....................(2)}$$

provided that $u>-1$, $u\neq 0$

In particular, taking $u=1$ and $u=-\frac{1}{2}$, we have

$$\tfrac{1}{2}<\operatorname{hyp}(2)<1 \text{ and } -1<\operatorname{hyp}(\tfrac{1}{2})<-\tfrac{1}{2}. \qquad\text{............(3)}$$

We shall now prove a fundamental property of $\operatorname{hyp}(t)$.

If n is any rational number,

$$\mathbf{hyp}\,(\mathbf{t^n})=\mathbf{n}\,\mathbf{hyp}\,(\mathbf{t}).$$

By definition, $\operatorname{hyp}(t^n)=\displaystyle\int_1^{t^n}\frac{1}{x}dx.$

Put $x=z^n$ so that $z=1$ when $x=1$, and $z=t$ when $x=t^n$; also $dx=nz^{n-1}\,dz$.

$\therefore$ $\operatorname{hyp}(t^n)=\displaystyle\int_1^t\frac{1}{z^n}\cdot nz^{n-1}\,dz=n\int_1^t\frac{1}{z}\,dz;$

$$\therefore\ \operatorname{hyp}(t^n)=n\operatorname{hyp}(t). \qquad\text{........................(4)}$$

In particular, putting $n=-1$, we have

$$\operatorname{hyp}\left(\frac{1}{t}\right)=-\operatorname{hyp}(t). \qquad\text{..........................(5)}$$

From (3), $\operatorname{hyp}(2)>\frac{1}{2}$, $\therefore$ $\operatorname{hyp}(2^n)=n\operatorname{hyp}(2)>\dfrac{n}{2}$ if $n>0$;

$$\therefore\ \text{if } x>2^n \text{ where } n>0,\ \operatorname{hyp}(x)>\operatorname{hyp}(2^n)>\frac{n}{2}.$$

$$\therefore\ \operatorname{hyp}(x)\to\infty \text{ when } x\to\infty.$$

Also, since $\operatorname{hyp}\left(\dfrac{1}{x}\right)=-\operatorname{hyp}(x)$,

$$\operatorname{hyp}(x)\to-\infty \text{ when } x\to 0,\ (x>0).$$

Thus hyp (x) *steadily increases* from $-\infty$ to $+\infty$ when x increases from 0 to ∞, and is zero when $x=1$. Its graph therefore takes the form shown in the figure. Also since from the definition it is continuous, we may conclude that it assumes once, and only once, any given value. In particular, there exists a *unique* value of x such that $\text{hyp}(x)=1$. This value is always denoted by e, so that e is defined by

$$\text{hyp}(e)\equiv\int_1^e \frac{1}{x}\,dx=1. \qquad \text{............(6)}$$

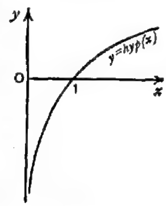

FIG. 3.

Since, from (3),

$$\text{hyp}(2)<1 \text{ and } \text{hyp}(4)=\text{hyp}(2^2)=2\,\text{hyp}(2)>1,$$

the value of e lies between 2 and 4. Actually e, like π, is irrational, and we shall find later that its value is $2{\cdot}71828\ldots$.

Natural Logarithms

For purposes of computation, numbers are expressed as powers of 10, and the corresponding index is the logarithm of the number to base 10. For mathematical purposes, it is much more convenient to express numbers as powers of e and the corresponding index is called the "natural" logarithm or the *Napierian* logarithm.

Thus, if $x=e^y$, y is called the natural logarithm of x, and we write $y=\log_e x$, or often, simply, $y=\log x$, if the context shows that the base is e. Throughout this chapter, the symbol $\log x$ will be used in this sense, the logarithm of x to base 10 being written $\log_{10} x$.

If $x=e^y$,

$$\text{hyp}(x)=\text{hyp}(e^y)=y\,\text{hyp}(e)=y.$$

But $y=\log_e x\equiv\log x$,

$$\therefore\ \mathbf{hyp}(\mathbf{x})\equiv\mathbf{log}\,\mathbf{x},\quad \mathbf{x}>\mathbf{0}.$$

We may therefore use the symbol "log" wherever hitherto we have used "hyp," and in future shall do so.

Thus the relations established above may be written :

From (1), $$\log t = \int_1^t \frac{1}{x}\,dx, \quad \text{..........(7)}$$
where $t > 0$.

From (2), $$\frac{u}{1+u} < \log(1+u) < u, \quad \text{..........(8)}$$
where $1+u > 0$, $u \neq 0$.

From (4), $$\log(t^n) = n \log t, \quad \text{..........(9)}$$
where n is any rational number and $t > 0$.

The properties obtained in *New Algebra*, Part III, Ch. IV, for logarithms to any base hold in particular for logarithms to base e and may be proved in the same way, although they can also be deduced directly from the integral form (see Ex. VI a, No. 17).

Thus $$\log(uv) = \log u + \log v\,;$$
$$\log\left(\frac{u}{v}\right) = \log u - \log v.$$

Also $$\log_v u = \frac{\log u}{\log v}.$$

Differentiation of log x

By definition,
$$\frac{d}{dx}\log x = \lim_{h\to 0}\frac{\log(x+h) - \log x}{h} = \lim_{h\to 0}\frac{1}{h}\log\left(1+\frac{h}{x}\right),$$
where we suppose that x and $x+h$ are positive.

But from (8), $$\frac{h}{x+h} < \log\left(1+\frac{h}{x}\right) < \frac{h}{x},$$
$$\therefore\ \frac{1}{h}\log\left(1+\frac{h}{x}\right) \text{ lies between } \frac{1}{x+h} \text{ and } \frac{1}{x}\,;$$
$$\therefore\ \frac{1}{h}\log\left(1+\frac{h}{x}\right) \to \frac{1}{x} \text{ when } h \to 0.$$
$$\therefore\ \frac{d}{dx}\log x = \frac{1}{x}, \quad \text{..........(10)}$$

EXERCISE VI. a

1. Draw, on squared paper, on a large scale, the part of the graph of $y = \frac{1}{x}$ from $x=1$ to $x=2$, and use it to find approximately the value of $\log_e 2$.

2. Figures 4 (i), (ii) represent the graph of $y=\frac{1}{x}$ between the ordinates $x=1$, $x=2$; the ordinate $x=1\frac{1}{2}$ is also drawn. What are the lengths of the ordinates and the areas of the rectangles shown? Deduce that $\frac{7}{12}<\log 2<\frac{5}{6}$.

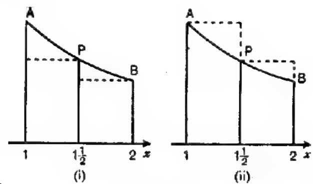

FIG. 4.

3. By taking the ordinates $x=1$, $1{\cdot}1$, $1{\cdot}2$, ..., $1{\cdot}9$, 2, show by the method of No. 2 that log 2 lies between

$$0{\cdot}1\left(1+\frac{1}{1{\cdot}1}+\frac{1}{1{\cdot}2}+\ldots+\frac{1}{1{\cdot}9}\right) \quad\text{and}\quad 0{\cdot}1\left(\frac{1}{1{\cdot}1}+\frac{1}{1{\cdot}2}+\ldots+\frac{1}{1{\cdot}9}+\frac{1}{2}\right)$$

and deduce that $0{\cdot}66<\log 2<0{\cdot}72$.

4. Given $\log 2 \simeq 0{\cdot}693$ and $\log 3 \simeq 1{\cdot}099$, evaluate

$$\text{(i)} \int_2^3 \frac{1}{x}\,dx\,; \quad \text{(ii)} \int_{20}^{30} \frac{1}{x}\,dx\,; \quad \text{(iii)} \int_1^8 \frac{1}{x}\,dx.$$

5. With the data of No. 4, evaluate

$$\text{(i)} \int_1^6 \frac{1}{x}\,dx\,; \quad \text{(ii)} \int_0^5 \frac{1}{1+z}\,dz\,; \quad \text{(iii)} \int_0^4 \frac{1}{1+2z}\,dz.$$

6. Find the values of

$$\text{(i)} \log e^2\,; \quad \text{(ii)} \log\frac{1}{e}\,; \quad \text{(iii)} \log\sqrt{e}.$$

7. Find x in terms of e if

(i) $\log x=3$; (ii) $\log x=-2$;
(iii) $\log x=1+\log 2$; (iv) $\log(\log x)=0$.

8. Simplify (i) $e^{\log x}$; (ii) $e^{x\log 2}$; (iii) $\log(ex)$.

9. Sketch the graphs of (i) $\log\left(\frac{1}{x}\right)$; (ii) $\log x^2$; (iii) $\log(2x)$

10. If $t>0$ and if $t\neq 1$, prove that

$$\frac{t-1}{t}<\log t<t-1.$$

11. What is the value of $\frac{d}{dx}\{\log(ax)-\log x\}$?

12. What is the value of $\frac{d}{dx}\log\frac{ax+b}{fx+g}$?

13. Find $\frac{d}{dx}(x\log x - x)$ and deduce the value of $\int_1^t \log x\, dx$.

14. Find the maximum value of $\frac{\log x}{x}$.

15. If p and q are positive, prove that

$$\log(p+q)-\log p > \frac{q}{p+q}.$$

16. Prove that $\log x < 2\sqrt{x}$.

17. Transform $\int_t^{ut}\frac{1}{x}\,dx$ by putting $x=tz$.

What result then follows from the relation,

$$\int_1^{ut}\frac{1}{x}\,dx=\int_1^{t}\frac{1}{x}\,dx+\int_t^{ut}\frac{1}{x}\,dx\,?$$

The Logarithmic Series

If a function $f(x)$ can be expanded as a power series of x,

$$a_0+a_1x+a_2x^2+\dots,\quad |x|<k,$$

a_0 is the value of $f(x)$ for $x=0$, and a_0+a_1x is its approximate value for a small, positive or negative, value of x. Thus the fact that $\log x$ is undefined when $x \leqslant 0$, and that $\log x \to -\infty$ when $x\to 0$, $x>0$, suggests that it cannot be expanded as a power series in x. But the function $\log(a+x)$ where $a>0$, which tends to $\log a$ when $x\to 0$, is capable of expansion in powers of x for a certain range of values of x. It is, however, sufficient to discuss the expansion of $\log(1+x)$, because

$$\log(a+x)=\log a+\log\left(1+\frac{x}{a}\right),\quad a>0.$$

If $|\mathbf{x}|<1$ or if $\mathbf{x}=1$, the sum to infinity of

$$\mathbf{x}-\tfrac{1}{2}\mathbf{x}^2+\tfrac{1}{3}\mathbf{x}^3-\dots+(-1)^{r-1}\frac{1}{r}\mathbf{x}^r+\dots$$

is $\log(1+\mathbf{x})$.

Denote the sum to n terms by $f_n(x)$.

Then $f_n(z)=z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\dots+(-1)^{n-1}\frac{1}{n}z^n$.

$$\therefore \frac{d}{dz}f_n(z)=1-z+z^2-\dots+(-1)^{n-1}z^{n-1}$$

$$=\frac{1-(-z)^n}{1-(-z)}=\frac{1}{1+z}+(-1)^{n+1}\frac{z^n}{1+z}.$$

But $\frac{d}{dz}\log(1+z)=\frac{1}{1+z}$ if $1+z>0$,

$$\therefore\ \frac{d}{dz}\{f_n(z)-\log(1+z)\}=(-1)^{n+1}\frac{z^n}{1+z}.$$

But, if z lies between 0 and x where $x>-1$, $\frac{1}{1+z}$ lies between 1 and $\frac{1}{1+x}$.

$\therefore\ \frac{d}{dz}\{f_n(z)-\log(1+z)\}$ lies between $(-1)^{n+1}z^n$ and $\frac{(-1)^{n+1}}{1+x}.z^n$.

Integrate from $z=0$ to $z=x$, where $x>-1$.

When $z=0$, $f_n(z)=0$ and $\log(1+z)=0$,

$$\therefore\ \text{if } x>-1,\ f_n(x)-\log(1+x)=(-1)^{n+1}\mathrm{A}\int_0^x z^n\,dz$$
$$=(-1)^{n+1}\frac{\mathrm{A}x^{n+1}}{n+1}$$

where A has some value between 1 and $\frac{1}{1+x}$.

But from p. 56, $\lim\limits_{n\to\infty} x^n=0$ if $|x|<1$

and obviously $\lim\limits_{n\to\infty}\frac{x^{n+1}}{n+1}=0$ if $x=1$.

$\therefore$ if $-1<x\leqslant 1$, $\lim\limits_{n\to\infty}\{f_n(x)-\log(1+x)\}=0$.

$$\therefore\ \lim_{n\to\infty} f_n(x)=\log(1+x).$$

$$\therefore\ \log(1+\mathbf{x})=\mathbf{x}-\tfrac{1}{2}\mathbf{x}^2+\tfrac{1}{3}\mathbf{x}^3-\dots \text{ if } -1<\mathbf{x}\leqslant 1. \quad\dots\dots\dots(11)$$

Note. (i) If $x=1$, this gives $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$, but this series converges so slowly that it is impracticable to use it for computing log 2.

(ii) The series is divergent when $x=-1$ (see p. 64), since it becomes $-(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots)$.

Writing $-x$ for x in (11), we have

$$\log(1-\mathbf{x})=-\mathbf{x}-\tfrac{1}{2}\mathbf{x}^2-\tfrac{1}{3}\mathbf{x}^3-\dots \text{ if } -1\leqslant\mathbf{x}<1. \quad\dots\dots(12)$$

Hence by subtraction,

$$\log(1+x)-\log(1-x)=2(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\dots),$$

or
$$\tfrac{1}{2}\log\frac{1+\mathbf{x}}{1-\mathbf{x}}=\mathbf{x}+\tfrac{1}{3}\mathbf{x}^3+\tfrac{1}{5}\mathbf{x}^5+\dots \text{ if } -1<\mathbf{x}<1. \quad\dots\dots\dots(13)$$

There are two useful alternative forms of (13).

(i) Put $\frac{1+x}{1-x}=y$ so that $x=\frac{y-1}{y+1}$.

Then $y>0$ corresponds to $-1<x<1$.

$$\therefore\ \tfrac{1}{2}\log y=\left(\frac{y-1}{y+1}\right)+\frac{1}{3}\left(\frac{y-1}{y+1}\right)^3+\frac{1}{5}\left(\frac{y-1}{y+1}\right)^5+\dots \text{ if } y>0. \quad \dots(14)$$

(ii) Put $\dfrac{1+x}{1-x}=\dfrac{p+1}{p}$ so that $x=\dfrac{1}{2p+1}$.

Then $p>0$ corresponds to $0<x<1$.

$$\therefore\ \log(p+1)-\log p=2\left\{\frac{1}{2p+1}+\frac{1}{3}\cdot\frac{1}{(2p+1)^3}+\dots\right\},\ p>0. \quad (15)$$

For an alternative method of setting out the proof of the expansion of $\log(1+x)$, see Durell and Robson, *Advanced Trigonometry*, pp. 84-85.

Example 1. Evaluate $\log 2$ to 5 places of decimals.

Put $y=2$ in the relation (14).

Then $\log 2=2\left(\dfrac{1}{3}+\dfrac{1}{3}\cdot\dfrac{1}{3^3}+\dfrac{1}{5}\cdot\dfrac{1}{3^5}+\dots\right)$.

$$\frac{1}{3}=0\cdot333{,}333{,}3.$$

$$\frac{1}{3^3}=0\cdot037{,}037{,}0;\qquad \frac{1}{3}\cdot\frac{1}{3^3}=0\cdot012{,}345{,}7.$$

$$\frac{1}{3^5}=0\cdot004{,}115{,}2;\qquad \frac{1}{5}\cdot\frac{1}{3^5}=0\cdot000{,}823{,}0.$$

$$\frac{1}{3^7}=0\cdot000{,}457{,}2;\qquad \frac{1}{7}\cdot\frac{1}{3^7}=0\cdot000{,}065{,}3.$$

$$\frac{1}{3^9}=0\cdot000{,}050{,}8;\qquad \frac{1}{9}\cdot\frac{1}{3^9}=0\cdot000{,}005{,}6.$$

$$\frac{1}{3^{11}}=0\cdot000{,}005{,}6;\qquad \frac{1}{11}\cdot\frac{1}{3^{11}}=0\cdot000{,}000{,}5.$$

$$\therefore\ \text{sum of first 6 terms}\simeq 2\times 0\cdot346{,}573{,}4$$
$$\simeq 0\cdot693{,}146{,}8.$$

$$\therefore\ \log 2=0\cdot69315 \text{ to 5 places of decimals.}$$

Degree of Accuracy

If, in Example 1, we had taken 5 terms only, the error in the value of $\frac{1}{2}\log 2$ would certainly have been less than

$$\frac{1}{11}\left(\frac{1}{3^{11}}+\frac{1}{3^{13}}+\frac{1}{3^{15}}+\dots\right).$$

and this is less than $\frac{1}{11}\cdot\frac{1}{3^{11}}\cdot\frac{1}{1-\frac{1}{9}}=\frac{9}{8}\cdot\frac{1}{11}\cdot\frac{1}{3^{11}}$, which is less than 0·000,000,6.

Therefore, for the required approximation, *five* terms are sufficient.

Example 2. Show how to calculate log 10.

Put $y=\frac{5}{4}$ in relation (14),

then $$\tfrac{1}{2}\log\frac{10}{8}=\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\frac{1}{5}\cdot\frac{1}{9^5}+\dots.$$

$$\therefore\ \log 10=3\log 2+2\left\{\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\dots\right\}.$$

Now make use of the value for log 2 obtained in Example 1. See also Ex. VI b, No. 5.

Example 3. *Find the coefficient of x^r in the expansion of $\log(1+x+x^2)$, given $|x|<1$.

$$\log(1+x+x^2)=\log\frac{1-x^3}{1-x}=\log(1-x^3)-\log(1-x).$$

But if $-1\leqslant x<1$,

$$\log(1-x^3)=-x^3-\tfrac{1}{2}x^6-\tfrac{1}{3}x^9-\dots,$$
$$\log(1-x)\ =-x\ -\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\dots.$$
$$\therefore\ \log(1+x+x^2)=(x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\dots)-(x^3+\tfrac{1}{2}x^6+\dots).$$

If r is not a multiple of 3, the coefficient of x^r is $\frac{1}{r}$; if r is a multiple of 3, the coefficient $=\frac{1}{r}-\frac{3}{r}=-\frac{2}{r}$.

Common Logarithms

If $\log_{10}N=x$, then $N=10^x$.

$$\therefore\ \log_e N=\log_e(10^x)=x\log_e 10.$$
$$\therefore\ \log_{10}N=\frac{\log_e N}{\log_e 10}.$$

Common logarithms, that is, logarithms to base 10, can therefore be calculated by multiplying the corresponding *natural* logarithm by $\frac{1}{\log_e 10}$; this constant factor is called the *modulus* of the common system, and may be obtained from Example 2; its value is 0·434,294,481,9 ..., and is denoted by μ.

Method of Proportional Parts

$$\log_{10}(\mathsf{N}+h)-\log_{10}\mathsf{N}=\log_{10}\frac{\mathsf{N}+h}{\mathsf{N}}=\mu\log_e\left(1+\frac{h}{\mathsf{N}}\right).$$

$\therefore$ if $\frac{h}{\mathsf{N}}$ is small, $\log_{10}(\mathsf{N}+h)-\log_{10}\mathsf{N}\simeq\frac{\mu h}{\mathsf{N}}$,

and if $\frac{1}{\mathsf{N}}$ is small, $\log_{10}(\mathsf{N}+1)-\log_{10}\mathsf{N}\simeq\frac{\mu}{\mathsf{N}}$;

$$\therefore \frac{\log_{10}(\mathsf{N}+h)-\log_{10}\mathsf{N}}{\log_{10}(\mathsf{N}+1)-\log_{10}\mathsf{N}}\simeq h.$$

$\therefore$ if the value of $\log_{10}(\mathsf{N}+1)-\log_{10}\mathsf{N}$ is given, where $\frac{1}{\mathsf{N}}$ is small and if $h<1$, the value of $\log_{10}(\mathsf{N}+h)-\log_{10}\mathsf{N}$ can be calculated by proportion. This is called the "method of proportional parts."

The error in taking $\frac{\mu}{\mathsf{N}}$ as the value of the difference

$\log_{10}(\mathsf{N}+1)-\log_{10}\mathsf{N}$ is less than $\frac{\mu}{2\mathsf{N}^2}$.

For example, 7-figure tables state that

$$\log_{10}34714=4{\cdot}540{,}504{,}7 \text{ and } \log_{10}34715=4{\cdot}540{,}517{,}2;$$

$$\therefore \log_{10}34715-\log_{10}34714=0{\cdot}000{,}012{,}5.$$

Suppose that the value of $\log_{10}34714{\cdot}48$ is required.

The proportional difference $=0{\cdot}48\times0{\cdot}000{,}012{,}5$

$=0{\cdot}000{,}006{,}0.$

$$\therefore \log_{10}34714{\cdot}48=4{\cdot}540{,}510{,}7.$$

In 7-figure tables, logarithms of all integers from 10,000 to 100,000 are given to 7 figures; therefore

$$\frac{h}{\mathsf{N}}<\frac{1}{10000}, \text{ also } \mu=0{\cdot}43\ldots<\tfrac{1}{2};$$

$\therefore$ the error in the "difference" computed by proportion is less than $\frac{1}{4\,.\,10^8}$, and so the computed difference is correct to 8 places of decimals.

EXERCISE VI. b

(As stated on p. 107, $\log x$ denotes $\log_e x$.)

1. Calculate $\log 1{\cdot}02$ to 4 places of decimals.

2. Use relation (15) to calculate $\log 3-\log 2$ correct to 5 places of decimals. Find the value of $\log 3$, given

$$\log 2\simeq 0{\cdot}693147.$$

Deduce the value of $\log_{10}3$, given $\log_{10}e=0{\cdot}43429$.

3. Given $2\cdot19722 < \log 9 < 2\cdot19723$, prove that
$$2\cdot39789 < \log 11 < 2\cdot39791.$$

4. Calculate $\log 13 - \log 12$ correct to 5 places of decimals. Then calculate $\log 13$ to 4 places of decimals, given that
$$\log 2 \simeq 0\cdot693147 \text{ and } \log 3 \simeq 1\cdot098612.$$

5. Prove that $\log \frac{128}{125} \simeq 0\cdot023716$. Hence show that
$$\log 10 \simeq 2\cdot302585,$$
given that $\log 2 \simeq 0\cdot6931472$.

6. If $a = \log \frac{6}{5}$, $b = \log \frac{10}{9}$, $c = \log \frac{25}{24}$, prove that
$$\log 2 = 3a + b + c.$$
Hence calculate $\log 2$ to 5 places of decimals.

Expand the following functions as power series in x, giving the coefficient of x^r and the conditions of validity:

7. $\log(1 - 3x)$. 8. $\log(1 + \frac{1}{4}x^2)$. 9. $\log(2 + x)$.

10. $\log(1 + x)^2$. 11. $\log\{(1 - x)(1 + 2x)\}$. 12. $\log(1 + 3x + 2x^2)$.

13. $(1 - x)\log(1 + x)$. 14. $(1 - x)^2 \log(1 - x)$.

15. $(1 + 2x)\log(1 - 2x^2)$. 16. $\log(1 - x^2 + x^4)$.

17. $\log \dfrac{\sqrt{(1 + x^2)}}{1 - x}$. 18. $\log \dfrac{1 - x + x^2}{1 + x}$.

19. $\log\left(1 + \dfrac{1}{x - 2}\right)$. 20. $\log(1 + x + x^2 + x^3 + x^4)$.

21. $\log(1 - 4x + 4x^2)$. 22. $\log\left(1 - \dfrac{2x^2}{1 - x}\right)$.

23. If $|x| > 1$, expand in powers of $\dfrac{1}{x}$,

(i) $\log(x + 1) - \log x$; (ii) $2\log x - \log(x + 1) - \log(x - 1)$.

24. If $|x| < 1$, expand in powers of x,
$$(1 + x)\log(1 + x) + (1 - x)\log(1 - x).$$

25. If α, β are the roots of $ax^2 + bx + c = 0$ where $\dfrac{b^2}{4a} > c > 0$, find in terms of α, β, the coefficient of x^r in the expansion of
$$\log(ax^2 + bx + c)$$
and state the conditions of validity.

Summation of Series

The sums to infinity of certain types of series can be obtained from the logarithmic expansion. The method of partial fractions is sometimes useful.

Example 4. Find the sum to infinity of

$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{9}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{9^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{9^3}+\dots.$$

The series may be written

$$\left(1+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\dots\right)+\left(\frac{1}{2}\cdot\frac{1}{9}+\frac{1}{4}\cdot\frac{1}{9^2}+\frac{1}{6}\cdot\frac{1}{9^3}+\dots\right)$$

$$=3[(\tfrac{1}{3})+\tfrac{1}{3}(\tfrac{1}{3})^3+\tfrac{1}{5}(\tfrac{1}{3})^5+\dots]+\tfrac{1}{2}[(\tfrac{1}{9})+\tfrac{1}{2}(\tfrac{1}{9})^2+\tfrac{1}{3}(\tfrac{1}{9})^3+\dots].$$

By relation (13), the sum to infinity of the series in the first bracket is $\frac{1}{2}\log\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}\log 2$; also the sum to infinity of the series in the second bracket is

$$-\log(1-\tfrac{1}{9})=-\log\tfrac{8}{9}=2\log 3-3\log 2.$$

$\therefore$ the sum to infinity of the given series is

$$3(\tfrac{1}{2}\log 2)+\tfrac{1}{2}(2\log 3-3\log 2)=\log 3.$$

Example 5. Find the sum to infinity of

$$\frac{1}{1.1.3}+\frac{1}{2.3.5}+\frac{1}{3.5.7}+\frac{1}{4.7.9}+\dots.$$

The nth term is $\frac{1}{n(2n-1)(2n+1)}$, and may be expressed in partial fractions in the form $\frac{1}{2n-1}-\frac{1}{n}+\frac{1}{2n+1}$, or, more conveniently, $\frac{1}{2n-1}-\frac{2}{2n}+\frac{1}{2n+1}$.

$\therefore$ the sum to n terms is

$$\left(\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)+\dots+\left(\frac{1}{2n-1}-\frac{2}{2n}+\frac{1}{2n+1}\right)$$

$$=2\left(1-\frac{1}{2}+\frac{1}{3}\dots+\frac{1}{2n-1}-\frac{1}{2n}\right)-\left(1-\frac{1}{2n+1}\right).$$

When $n\to\infty$, the limits of the two brackets are $\log 2$ and 1;

$\therefore$ the sum to infinity is $2\log 2-1$.

EXERCISE VI. c

Sum to infinity the following series:

1. $\frac{1}{1.2}+\frac{1}{2.2^2}+\frac{1}{3.2^3}+\dots.$

2. $\frac{1}{1.3}-\frac{1}{2.3^2}+\frac{1}{3.3^3}-\dots.$

3. $\frac{1}{1.4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\dots.$

4. $\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots,\ |x|<1.$

5. $\dfrac{2xy}{x^2+y^2}+\dfrac{1}{3}\left(\dfrac{2xy}{x^2+y^2}\right)^3+\dfrac{1}{5}\left(\dfrac{2xy}{x^2+y^2}\right)^5+\dots,\quad x<y.$

6. $\dfrac{1}{1.2}+\dfrac{1}{3.4}+\dfrac{1}{5.6}+\dots.$ 7. $\dfrac{1}{1.3}+\dfrac{1}{2.5}+\dfrac{1}{3.7}+\dots.$

8. $\dfrac{1}{1.2}-\dfrac{1}{2.3}+\dfrac{1}{3.4}-\dots.$ 9. $\dfrac{1}{1.2.3}+\dfrac{1}{3.4.5}+\dfrac{1}{5.6.7}+\dots.$

10. $\dfrac{x}{1.2}+\dfrac{x^2}{2.3}+\dfrac{x^3}{3.4}+\dots,\quad |x|<1.$

11. $\dfrac{x^2}{1.3}+\dfrac{x^4}{3.5}+\dfrac{x^6}{5.7}+\dots,\quad |x|<1.$

12. $\dfrac{4x}{1.3}+\dfrac{6x^2}{2.4}+\dfrac{8x^3}{3.5}+\dots\quad |x|<1.$

13. $1+\left(\dfrac{1}{2}+\dfrac{1}{3}\right)\dfrac{1}{4}+\left(\dfrac{1}{4}+\dfrac{1}{5}\right)\dfrac{1}{4^2}+\dots.$

14. $\dfrac{5}{1.2.3}+\dfrac{6}{2.3.4}+\dfrac{7}{3.4.5}+\dots.$

15. $\dfrac{1}{1.2.3}+\dfrac{1}{5.6.7}+\dfrac{1}{9.10.11}+\dots.$

16. $\dfrac{3}{2.1.2}-\dfrac{4}{2^2.2.3}+\dfrac{5}{2^3.3.4}-\dots.$

17. Prove that

$$\left(\frac{1}{3}-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{2^2}\right)+\frac{1}{3}\left(\frac{1}{3^3}-\frac{1}{2^3}\right)+\frac{1}{4}\left(\frac{1}{3^4}+\frac{1}{2^4}\right)+\dots=0.$$

18. Prove that under certain conditions (to be stated)

$$\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\dots=\frac{2}{y}+\frac{2}{3y^3}+\frac{2}{5y^5}+\dots$$

where $y=2x^2-1$.

19. If $0<x<1$ and if $y=\dfrac{x}{1+x}$, express the sum to infinity $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots$ as a power series in y.

20. If $|x|>2$, prove that the sum to infinity of

$$\frac{2}{x^3-3x}+\frac{2^3}{3(x^3-3x)^3}+\frac{2^5}{5(x^3-3x)^5}+$$

is $\log\dfrac{x-1}{x+1}+\dfrac{1}{2}\log\dfrac{x+2}{x-2}$.

21. If $x>0$, find the sum to infinity of

$$\frac{x-1}{x+1}+\frac{x^2-1}{2(x+1)^2}+\frac{x^3-1}{3(x+1)^3}+\dots.$$

22. What is the coefficient of x^n in the expansion

$$\log(1-2x+x^2)=\log[1-x(2-x)]$$
$$=-x(2-x)-\tfrac{1}{2}x^2(2-x)^2-\tfrac{1}{3}x^3(2-x)^3-\dots.$$

By using the fact that this is the same as the coefficient of x^n in $2\log(1-x)$, prove that

$$2^n-n\,.\,2^{n-2}+\frac{n(n-3)}{1\,.\,2}2^{n-4}-\frac{n(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-6}+\dots=2.$$

23. Assuming that the coefficients of x^{3n} may be equated when the two sides of the identity $\log(1+x^3)\equiv\log(1+x)+\log(1-x+x^2)$ are expanded in powers of x, find the sum of the series

$$1-\frac{3n-3}{2!}+\frac{(3n-4)(3n-5)}{3!}-\frac{(3n-5)(3n-6)(3n-7)}{4!}+\dots.$$

Approximations

Some indication has been given in Example 1, p. 112, of the size of the error due to replacing the logarithmic function by a few leading terms of its expansion.

If $0<x\leqslant 1$, the terms of the series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$

are alternately positive and negative, and steadily decrease in numerical value. Consequently $\log(1+x)$ is greater than the sum of $2p$ terms and less than the sum of $(2p+1)$ terms, where p is integral, so that the error in replacing $\log(1+x)$ by the first n terms of the expansion is less than $\dfrac{x^{n+1}}{n+1}$.

If $-1<x<0$, the terms of the same series are all negative and the error is then certainly less than the numerical value of

$$\frac{x^{n+1}}{n+1}(1-x+x^2-\dots),$$

or less than the numerical value of $\dfrac{x^{n+1}}{(n+1)(1+x)}$. These results also follow from the proof of the expansion on p. 111.

Similarly, if $\tfrac{1}{2}\log\dfrac{1+x}{1-x}$, where $|x|<1$, is replaced by the first n terms of the series

$$x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\dots,$$

the error is less than the numerical value of $\dfrac{x^{2n+1}}{(2n+1)(1-x^2)}$.

Example 6. Show that the error in Napier's formula,

$$\log\frac{a}{b} \simeq \tfrac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right),$$

where $a > b > 0$ and $a-b$ is small compared with a, is approximately $\dfrac{(a-b)^3}{6a^3}$ and is certainly less than $\dfrac{(a-b)^3(3a-2b)}{6a^3b}$.

Put $a-b=x$ so that $\dfrac{x}{a}$ is small and positive.

$$\log\frac{a}{b} = -\log\frac{b}{a} = -\log\left(1-\frac{x}{a}\right);$$

$$\therefore \log\frac{a}{b} \simeq \frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{3a^3}. \quad \text{.........................(i)}$$

$$\tfrac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{x}{2}\left(\frac{1}{a}+\frac{1}{a-x}\right)=\frac{x}{2a}\left\{1+\left(1-\frac{x}{a}\right)^{-1}\right\};$$

$$\therefore \tfrac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right)\simeq\frac{x}{2a}\left\{1+1+\frac{x}{a}+\frac{x^2}{a^2}\right\}; \quad \text{.............(ii)}$$

$$\therefore \tfrac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right)-\log\frac{a}{b}\simeq\frac{x^3}{2a^3}-\frac{x^3}{3a^3}=\frac{x^3}{6a^3}\simeq\frac{(a-b)^3}{6a^3}.$$

Since $\dfrac{x}{a}$ is positive, the approximation in (i) is less than the true value, but the error

$$<\frac{x^4}{4a^4}\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\dots\right)=\frac{x^4}{4a^4}\cdot\frac{1}{1-\dfrac{x}{a}}=\frac{x^4}{4a^3(a-x)};$$

the approximation in (ii) is also less than the true value and the error is $\dfrac{x}{2a}\cdot\dfrac{x^3}{a^3}\left(1+\dfrac{x}{a}+\dfrac{x^2}{a^2}+\dots\right)=\dfrac{x^4}{2a^3(a-x)}$.

Since both are errors of defect, the expression

$$\tfrac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right)-\log\frac{a}{b}$$

is less than

$$\frac{x^3}{6a^3}+\frac{x^4}{2a^3(a-x)}=\frac{x^3}{6a^3}\left(1+\frac{3x}{a-x}\right)=\frac{(a-b)^3(3a-2b)}{6a^3b}.$$

Example 7. Find the value of $\lim\limits_{x \to 1} \dfrac{1 - x + \log x}{1 - \sqrt{(2x - x^2)}}$.

Put $1 - x = h$ and suppose that h is small.

Then $2x - x^2 = 1 - (1 - x)^2 = 1 - h^2$;

$$\therefore \text{ the function} = \frac{h + \log(1 - h)}{1 - \sqrt{(1 - h^2)}}$$

$$= \frac{\{h + \log(1 - h)\}\{1 + \sqrt{(1 - h^2)}\}}{1 - (1 - h^2)}$$

$$= \frac{(h - h - \frac{1}{2}h^2 - ah^3)(1 + 1 - bh^2)}{h^2}$$

where, as in Example 6, it can be shown that

$$0 < a < \frac{1}{3(1 - |h|)} \quad \text{and} \quad 0 < b < \frac{1}{2(1 - h^2)};$$

$\therefore$ the function $= -1 - 2ah + \frac{1}{2}bh^2 + abh^3$.

$\therefore$ when $x \to 1$, since $h \to 0$, the function $\to -1$.

EXERCISE VI. d

1 If x is small, prove that

$$\log(1 + x^2) - \log(1 + x) - \log(1 - x) \simeq 2x^2 + \tfrac{2}{3}x^6.$$

2. If x is large, prove that $\log(x + 1) - \log x \simeq \dfrac{1}{x}$.

3. If x is small, find the best approximation for $\log(1 + x)$ which can be expressed in the form

$$\text{(i) } \frac{ax}{1 + bx}; \quad \text{(ii) } \frac{x(1 + ax)}{1 + bx + cx^2}.$$

4. If x is small, prove that

$$x - \sqrt{(1 + x)} \,.\, \log(1 + x) \simeq \frac{x^3}{24}.$$

Evaluate the following limits:

5. $\lim\limits_{x \to 1} \dfrac{\log x}{x - 1}$.

6. $\lim\limits_{x \to 0} \dfrac{x - \log(1 + x)}{x^2}$.

7. $\lim\limits_{x \to 0} \dfrac{\log(1 + x)}{(1 + x)^m - (1 + x)^{-m}}$.

8. $\lim\limits_{x \to 1} \dfrac{\log_{10} x}{x - 1}$.

9. $\lim\limits_{x \to 0} \dfrac{1}{x} \log\{\log(1 + x)^{\frac{1}{x}}\}$.

10. $\lim\limits_{x \to 1} \left(\dfrac{x}{x - 1} - \dfrac{1}{\log x}\right)$.

11. $\lim\limits_{x \to \infty} \sqrt{(x^2 + x)} \,.\, \{\log(x + 1) - \log x\}$.

12. $\lim\limits_{x \to 0} \dfrac{1}{x^2} \{(1 + x)\log(1 + x) + (1 - x)\log(1 - x)\}$.

13. If n is a positive integer >1, prove that $\log n$ lies between $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}$ and $\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

Prove also that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$ decreases as n increases. [From this, it can be inferred that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$ tends to a limit when $n\to\infty$; this limit is called Euler's constant and is denoted by γ, the value of γ is $0{\cdot}577\ \ldots$.]

14. Use the statement in No. 13 to prove that

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right)=\log 2.$$

15. Use the statement in No. 13 to prove that

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{np}\right)=\log p$$

where n and p are positive integers.

The Function e^x

If x is a fraction with an even denominator, e^x has two values, numerically equal but opposite in sign. But throughout this chapter, we shall use e^x to denote the *positive* value only.

Thus if $e^x=y$, then $x=\log y$.

Since $\log y$ steadily increases from $-\infty$ to $+\infty$ when y increases from 0 to ∞, it follows that e^x steadily increases from 0 to ∞ when x increases from $-\infty$ to $+\infty$; that is, $e^{x_1}>e^{x_2}>0$ whenever $x_1>x_2$.

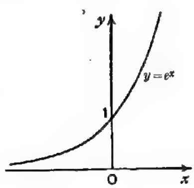

FIG. 5.

The graph of $y=e^x$ is of course the same as the graph of $x=\log y$, and is therefore obtained from the graph on p. 107 by interchanging the axes of x and y, or, equally well, by taking the image of $y=\log x$ in the line $y=x$; it therefore assumes the form shown in the figure.

Also $\quad \dfrac{dx}{dy}=\dfrac{d}{dy}(\log y)=\dfrac{1}{y}; \quad \therefore\ \dfrac{dy}{dx}=y; \quad \therefore\ \dfrac{d}{dx}(e^x)=e^x.$

Therefore also $\qquad \int e^x\,dx=e^x.$

Expansion of e^x in Powers of x

If we assume that e^x can be expanded in the form

$$e^x = a_0 + a_1 x + a_2 x^2 + \dots + a_r x^r + \dots ;$$

and if we also assume that

$$\frac{d}{dx}(e^x) = \frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \dots + \frac{d}{dx}(a_r x^r) + \dots ,$$

and that we may continue to differentiate in this way, it is easy to find the values of $a_0, a_1, a_2, \dots$.

Putting $x = 0$ in the first equation, we have $1 = a_0$.

The second equation is

$$e^x = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots ,$$

and continuing the process we have

$$e^x = 1 . 2a_2 + 2 . 3a_3 x + 3 . 4a_4 x^2 + \dots ,$$

$$e^x = 1 . 2 . 3a_3 + 2 . 3 . 4a_4 x + \dots ,$$

and so on.

Putting $x = 0$ in these equations in turn, we have

$$1 = a_1 ; \quad 1 = 1 . 2a_2 ; \quad 1 = 1 . 2 . 3a_3 ; \dots .$$

$$\therefore \; a_0 = 1 ; \quad a_1 = 1 ; \quad a_2 = \frac{1}{2!} ; \quad a_3 = \frac{1}{3!} ; \dots .$$

Therefore the expansion is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots .$$

This expansion is called the "exponential series," and it was proved on p. 75 that it is absolutely convergent for all values of x. *But the assumptions stated above are not easy to justify.* A valid proof, however, can be formulated by the same method as was used for the expansions of $(1 + x)^m$ and $\log(1 + x)$.

The Exponential Theorem

The sum to infinity of the series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots$$

is e^x, for all values of x.

Denote the sum to n terms of the series by $f_n(x)$.

Then
$$f_n(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^{n-1}}{(n-1)!} ;$$

$$\therefore \; \frac{d}{dz} f_n(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^{n-2}}{(n-2)!} ;$$

$$\therefore\ f_n(z) - \frac{d}{dz}f_n(z) = \frac{z^{n-1}}{(n-1)!};$$

$$\therefore\ \frac{d}{dz}\{f_n(z)e^{-z}\} = e^{-z}\frac{d}{dz}f_n(z) - f_n(z)e^{-z}$$

$$= -e^{-z}\{f_n(z) - \frac{d}{dz}f_n(z)\} = -\frac{e^{-z}z^{n-1}}{(n-1)!}.$$

But if z lies between 0 and x, e^{-z} lies between 1 and e^{-x};

$$\therefore\ \frac{d}{dz}\{f_n(z)e^{-z}\} \text{ lies between } -\frac{z^{n-1}}{(n-1)!} \text{ and } -\frac{e^{-x}z^{n-1}}{(n-1)!}.$$

Integrate from $z=0$ to $z=x$.

When $z=0$, $f_n(z)=1$ and $e^{-z}=1$,

$$\therefore\ f_n(x)e^{-x} - 1 = -\mathsf{A}\int_0^x \frac{z^{n-1}}{(n-1)!}\,dz = -\mathsf{A}\frac{x^n}{n!},$$

where A has some value between 1 and e^{-x}.

Multiply each side by e^x,

$$\therefore\ f_n(x) - e^x = -\mathsf{B}\frac{x^n}{n!},$$

where B has some value between e^x and 1.

But from p. 76, $\lim_{n\to\infty}\dfrac{x^n}{n!} = 0$ for all values of x;

$$\therefore\ \lim_{n\to\infty} f_n(x) = e^x \text{ for all values of } x.$$

For an alternative method of proof of the expansion of e^x, see Durell and Robson, *Advanced Trigonometry*, p. 97.

The Value of e

Putting $x=1$ in the expansion of e^x, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots .$$

This series converges rapidly, and successive terms are easily computed by continued division. Thus

$1+1$	=	2·000 000 0
$1/2!$	=	0·500 000 0
$1/3!$	=	0·166 666 7
$1/4!$	=	0·041 666 7
$1/5!$	=	0·008 333 3
$1/6!$	=	0·001 388 9
$1/7!$	=	0·000 198 4
$1/8!$	=	0·000 024 8
$1/9!$	=	0·000 002 8
		2·718 281 6

The error in taking only the first 10 terms is less than

$$\frac{1}{10!}\left(1+\frac{1}{10}+\frac{1}{10^2}+\dots\right)$$

or less than
$$\frac{1}{10!}\cdot\frac{1}{1-\frac{1}{10}}$$

or less than
$$\frac{1}{9!\,9}<0{\cdot}000{,}000{,}4.$$

Therefore this calculation certainly gives e correct to 5 places of decimals; actually $e=2{\cdot}7182818\ \dots$, and therefore the result is correct to 6 places.

Similarly, e exceeds $1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{(n-1)!}$ by

$$\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\dots$$

and this is less than
$$\frac{1}{n!}\left\{1+\frac{1}{n}+\frac{1}{n^2}+\dots\right\}$$

or less than
$$\frac{1}{n!}\cdot\frac{1}{1-\dfrac{1}{n}}=\frac{1}{(n-1)!(n-1)}.$$

$$\therefore\ 1+\frac{1}{1!}+\dots+\frac{1}{(n-1)!}<e<1+\frac{1}{1!}+\dots+\frac{1}{(n-1)!}+\frac{1}{(n-1)!(n-1)}.$$

To prove that e is not rational

If possible, suppose $e=\frac{p}{q}$ where p, q are integers. Then $\frac{p}{q}$ lies between $1+\frac{1}{1!}+\dots+\frac{1}{q!}$ and $1+\frac{1}{1!}+\dots+\frac{1}{q!}+\frac{1}{q!\,q}$.

$\therefore$ multiplying by $q!$, it follows that the integer $(q-1)!\,p$ lies between K and $K+\frac{1}{q}$, where K is the integer, $q!+\frac{q!}{1!}+\dots+\frac{q!}{q!}$.

But this is impossible, because there is no integer between K and $K+\frac{1}{q}$ since K is itself an integer.

$\therefore$ e cannot be expressed as the ratio of two integers.

Remainder after n Terms of the Exponential Series

The error in replacing e^x by

$$1+x+\frac{x^2}{2!}+\dots+\frac{x^{n-1}}{(n-1)!}$$

is
$$\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}+\dots,$$
and this is often called the "remainder after n terms."

If $x>0$, this is less than $\frac{x^n}{n!}\left(1+\frac{x}{n}+\frac{x^2}{n^2}+\dots\right)$ and for $n>x$, this is less than $\frac{x^n}{n!}\cdot\frac{1}{1-\frac{x}{n}}=\frac{x^n}{(n-1)!(n-x)}$.

If $x<0$, the terms are alternately positive and negative and decrease numerically after the rth term if $r>-x$.

$\therefore$ the remainder is numerically less than $\frac{x^n}{n!}$ if $n>-x$.

Expansion of a^x, $a>0$, in Powers of x

If $\log a=m$, a being positive, then $a=e^m$;
$$\therefore\ a^x=e^{mx}=1+mx+\frac{m^2x^2}{2!}+\frac{m^3x^3}{3!}+\dots;$$
$$\therefore\ \mathbf{a^x=1+x\log a+\frac{x^2(\log a)^2}{2!}+\frac{x^3(\log a)^3}{3!}+\dots,\ a>0,}$$
for all values of x.

Example 8. Expand in powers of x
$$\text{(i) } \frac{e^x+e^{-x}}{2};\quad \text{(ii) } \frac{e^x-e^{-x}}{2}.$$
$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots.$$
Writing $-x$ for x, we have
$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots.$$
$$\therefore\ \tfrac{1}{2}(e^x+e^{-x})=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$$
and
$$\tfrac{1}{2}(e^x-e^{-x})=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots.$$

Example 9. Find the coefficient of x^n in the expansion of
$$\frac{1-3x+x^2}{e^x}.$$
$$\frac{1-3x+x^2}{e^x}=(1-3x+x^2)e^{-x}$$
$$=(1-3x+x^2)\left(1-x+\frac{x^2}{2!}-\dots+(-1)^n\frac{x^n}{n!}+\dots\right).$$

$\therefore$ the coefficient of x^n, if $n > 1$, is

$$(-1)^n\left\{\frac{1}{n!}+\frac{3}{(n-1)!}+\frac{1}{(n-2)!}\right\}$$

$$=(-1)^n\cdot\frac{1}{n!}\{1+3n+n(n-1)\}=(-1)^n\frac{(n+1)^2}{n!}.$$

This form is also true for $n=0$, $n=1$, because the first two terms are $1-4x$.

Example 10. Find the sum to infinity of

$$\frac{2^2+1}{2!}+\frac{2(3^2+1)}{3!}+\frac{3(4^2+1)}{4!}+\ldots.$$

The nth term $=\dfrac{n\{(n+1)^2+1\}}{(n+1)!}=\dfrac{n(n+1)^2+n}{(n+1)!}.$

But $n(n+1)^2+n=(n+1)(n^2+n)+n$

$$=(n+1)n(n-1)+2n(n+1)+(n+1)-1.$$

$$\therefore \text{ the } n\text{th term} =\frac{(n+1)n(n-1)}{(n+1)!}+\frac{2n(n+1)}{(n+1)!}+\frac{n+1}{(n+1)!}-\frac{1}{(n+1)!}$$

$$=\frac{1}{(n-2)!}+\frac{2}{(n-1)!}+\frac{1}{n!}-\frac{1}{(n+1)!} \text{ for } n>2.$$

By inspection, 1st term $=2+\dfrac{1}{1!}-\dfrac{1}{2!}$

and 2nd term $=1+\dfrac{2}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}.$

By the formula, 3rd term $=\dfrac{1}{1!}+\dfrac{2}{2!}+\dfrac{1}{3!}-\dfrac{1}{4!}$;

and so on.

$\therefore$ the sum to n terms, for $n>1$,

$$=\left\{1+\frac{1}{1!}+\ldots+\frac{1}{(n-2)!}\right\}+2\left\{1+\frac{1}{1!}+\ldots+\frac{1}{(n-1)!}\right\}+\left\{\frac{1}{1!}-\frac{1}{(n+1)!}\right\}.$$

But when $n\to\infty$, these brackets tend respectively to e, e, 1.

$\therefore$ the sum to infinity $=e+2e+1=3e+1.$

EXERCISE VI. e

1. Write down series whose sums to infinity are

(i) e^2; (ii) $e+\dfrac{1}{e}$; (iii) $\dfrac{1}{e^2}$; (iv) $\left(e-\dfrac{1}{e}\right)^2$.

2. Calculate $\sqrt{e}$ correct to 5 places of decimals.

3. Calculate $\sqrt[10]{e}$ correct to 7 places of decimals.

Express as power series in x:

4. $(1-x)e^x$. 5. $\dfrac{1+x}{e^x}$. 6. 10^x.

Find the coefficient of x^n in the expansions of the functions in Nos. 7-11.

7. $\dfrac{e^{5x}+e^x}{e^{3x}}$. 8. e^{2x+3}. 9. $\dfrac{x^2-3x+1}{e^x}$.

10. $\left(\dfrac{a+bx}{e^x}\right)^2$. 11. $\sqrt{\left(\dfrac{x^2-4x+4}{e^x}\right)}$.

12. Simplify $\left(\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+\ldots\right)\div\left(1+\dfrac{1}{3!}+\dfrac{1}{5!}+\ldots\right)$.

Find the sums to infinity of the following series:

13. $\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}-\ldots$. 14. $3+\dfrac{3^2}{2!}+\dfrac{3^3}{3!}+\ldots$.

15. $1-\dfrac{4}{2!}+\dfrac{4^2}{3!}-\dfrac{4^3}{4!}+\ldots$. 16. $1+\dfrac{4^2}{3!}+\dfrac{4^4}{5!}+\ldots$.

17. $1+\dfrac{2}{1!}+\dfrac{3}{2!}+\dfrac{4}{3!}+\ldots$. 18. $\dfrac{1^2}{2!}+\dfrac{2^2}{3!}+\dfrac{3^2}{4!}+\ldots$.

19. $\dfrac{1}{3!}+\dfrac{2}{5!}+\dfrac{3}{7!}+\ldots$. 20. $\dfrac{1}{1!}+\dfrac{1+2}{2!}+\dfrac{1+2+3}{3!}+\ldots$.

21. $\dfrac{1}{1!}+\dfrac{1+2}{2!}+\dfrac{1+2+2^2}{3!}+\ldots$. 22. $1+\dfrac{3}{2}+\dfrac{5}{2 \cdot 4}+\dfrac{7}{2 \cdot 4 \cdot 6}+\ldots$.

23. $\dfrac{1^2}{2!}-\dfrac{2^2}{3!}+\dfrac{3^2}{4!}-\ldots$. 24. $1+\dfrac{2^3}{2!}+\dfrac{3^3}{3!}+\ldots$.

25. $\dfrac{2 \cdot 5}{1!}+\dfrac{3 \cdot 7}{2!}+\dfrac{4 \cdot 9}{3!}+\ldots$. 26. $\dfrac{1^2}{3!}+\dfrac{2^2}{5!}+\dfrac{3^2}{7!}+\ldots$.

Find the sums to infinity of the series whose rth terms are as follows:

27. $\dfrac{x^r}{(r+1)!}$. 28. $\dfrac{x^r}{(2r-1)!}$, $(x>0)$. 29. $\dfrac{3r}{(2r-1)!}$.

30. $(-1)^r\dfrac{(\log 3)^r}{r!}$. 31. $\dfrac{(\log x)^{2r}}{(2r)!}$. 32. $\dfrac{x^r}{(r+2) \cdot r!}$.

33. Find the coefficients of x^n and x^{n+1} in $(e^x-1)^n$.

34. Find the coefficient of x^r in the series

$$\frac{x+1}{1!}+\frac{(x+1)^2}{2!}+\frac{(x+1)^3}{3!}+\ldots.$$

35. If n is a positive integer, find the coefficient of x^4 in

$$(e^x + e^{-x})^n.$$

36. By expanding $(e^x - 1)^n$ in two ways, n being a positive integer, and equating coefficients of x^n, prove that

$$n^n - n(n-1)^n + \frac{n(n-1)}{1 \cdot 2}(n-2)^n - \dots,\ n \text{ terms},\ = n!.$$

37. Use the method of No. 36 to prove that

$$\text{(i) } n^{n-1} - n(n-1)^{n-1} + \frac{n(n-1)}{1 \cdot 2}(n-2)^{n-1} - \dots,$$
$$n \text{ terms},\ = 0;$$

$$\text{(ii) } n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1 \cdot 2}(n-2)^{n+1} - \dots,$$
$$n \text{ terms},\ = \tfrac{1}{2}n \cdot (n+1)!.$$

38. By expanding $(e^x - e^{-x})^n$ in two ways, n being a positive integer, prove that

$$n^n - n(n-2)^n + \frac{n(n-1)}{1 \cdot 2}(n-4)^n - \dots,\ (n+1) \text{ terms},\ = 2^n \cdot n!.$$

The Exponential Limit

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x \text{ for all values of } x.$$

If n and $n + x$ are positive, from relation (8), p. 108,

$$\frac{x}{n+x} < \log\left(1 + \frac{x}{n}\right) < \frac{x}{n}.$$

$$\therefore\ \frac{nx}{n+x} < n\log\left(1 + \frac{x}{n}\right) = \log\left(1 + \frac{x}{n}\right)^n < x.$$

But when $n \to \infty$, $\quad \dfrac{nx}{n+x} = \dfrac{x}{1 + \dfrac{x}{n}} \to x.$

$$\therefore\ \lim_{n\to\infty}\log\left(1 + \frac{x}{n}\right)^n = x;$$

$$\therefore\ \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

In particular, putting $x = 1$, we have

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e,$$

and writing $-x$ for x we have

$$\lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n = e^{-x}.$$

Example 11. If x is small, expand $(1+x)^{\frac{1}{x}}$ in powers of x, as far as x^2.

$$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)} \simeq e^{1-\frac{1}{2}x+\frac{1}{3}x^2}$$
$$\simeq e \,.\, e^{-\frac{1}{2}x+\frac{1}{3}x^2}$$
$$\simeq e\left\{1-(\tfrac{1}{2}x-\tfrac{1}{3}x^2)+\frac{1}{2!}(\tfrac{1}{2}x-\tfrac{1}{3}x^2)^2\right\}$$
$$\simeq e\{1-\tfrac{1}{2}x+\tfrac{1}{3}x^2+\tfrac{1}{8}x^2\};$$
$$\therefore\ (1+x)^{\frac{1}{x}} \simeq e(1-\tfrac{1}{2}x+\tfrac{11}{24}x^2).$$

EXERCISE VI. f

1. If x is small, prove that $\dfrac{1}{1+x}-e^{-x} \simeq \frac{1}{2}x^2-\frac{5}{6}x^3$.

2. Expand $\dfrac{x}{e^x-1}$ in powers of x as far as x^4.

3. If x is small, prove that $e^{2x(1-x)}+e^{-2x(1+x)} \simeq 2$; and find to what order this approximation holds.

4. If x is so small that x^4 is negligible, prove that
$$\log\frac{1+e^x}{2} = \tfrac{1}{2}x+\tfrac{1}{8}x^2.$$

5. Evaluate $\lim\limits_{x\to 0}\dfrac{1}{x}(e^{ax}-e^{bx})$.

6. Evaluate $\lim\limits_{x\to 1}\dfrac{e^{x-1}+e^{1-x}-2}{(1-x)^2}$.

7. If $a>0$, prove that $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a$.

8. Prove that the remainder after n terms of the series, $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dots$, lies between $\left(1+\dfrac{1}{n+1}\right)\dfrac{1}{n!}$ and $\left(1+\dfrac{1}{n}\right)\dfrac{1}{n!}$.

9. If n is a positive integer, use the binomial theorem to prove that (i) $\left(1+\dfrac{1}{n}\right)^n$ increases as n increases, (ii) $\left(1+\dfrac{1}{n}\right)^n<3$. What follows from these facts?

10. If x is small, prove that
$$(1+x)^{1+x} \simeq 1+x+x^2+\tfrac{1}{2}x^3.$$

11. If p is small and $a>0$, prove that a root of the equation $x^{2+p}=a^2$ is $a(1-\frac{1}{2}p\log a)$ approximately.

12. Simplify $e^{\log(1+x)}$. If x is small, prove that $1+x$ exceeds $e^{x-\frac{1}{2}x^2}$ by $\frac{1}{3}x^3$ approximately.

13. If x is large, prove that $\left(1+\dfrac{1}{x}\right)^{x+1} \simeq e\left(1+\dfrac{1}{2x}\right)$, and find a closer approximation.

14. Prove that $\lim\limits_{x\to 1} \dfrac{\log x - x^x + 1}{\log x - x + 1} = 3$.

15. Prove that $\lim\limits_{x\to 0} \dfrac{1}{x} \log \dfrac{xe^x}{e^x - 1} = \frac{1}{2}$.

16. Prove that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n} + \dfrac{1}{n^2}\right)^n = e$.

17. If p is small, prove that a root of the equation $e^x + x = 1 + p$ is $\frac{1}{2}p - \dfrac{p^2}{16}$ approximately.

18. If ${}_n\mathsf{P}_r$ has its usual meaning, prove that $1 + \sum\limits_{r=1}^{n} {}_n\mathsf{P}_r$ equals the integer next less than $e(n!)$.

19. If p is small, prove that a root of the equation

$$x \log x + x = 1 + p$$

is $1 + \frac{1}{2}p$ approximately, and find a closer approximation.

20. If n is a positive integer and if $x > 0$, prove by differentiation that $\dfrac{(n+1+x)^{n+1}}{(n+x)^n}$ increases as x increases.

Hence show that $\left(1+\dfrac{x}{n}\right)^n < \left(1+\dfrac{x}{n+1}\right)^{n+1}$.

TEST PAPERS A. 11–20

A. 11

1. If $\dfrac{\log a}{p} = \dfrac{\log b}{q} = \dfrac{\log c}{r} = \log x$, express $\dfrac{b^2}{ac}$ as a power of x.

2. In how many ways can 3 dots and 5 dashes be arranged in a row (i) if there is no restriction, (ii) if the dots are all separated ?

3. Find the sum of the series

(i) $(1\frac{1}{2})^2 + (2\frac{1}{2})^2 + (3\frac{1}{2})^2 + \dots$ to n terms ;

(ii) $\dfrac{1}{2n^2-1} + \dfrac{1}{3(2n^2-1)^3} + \dfrac{1}{5(2n^2-1)^5} + \dots$ to infinity, if $|n| > 1$.

4. Express in partial fractions

$$\text{(i)}\ \frac{10x}{(x-1)(x-3)(x+4)}\,; \quad \text{(ii)}\ \frac{x^3}{(x-1)^5}.$$

5. (i) Find which is the greatest term in the expansion of $(1+6x)^{10}$ when $x=\frac{1}{5}$.

(ii) What is the coefficient of x^n in the expansion of

$$(1-x)e^{1+x}\ ?$$

A. 12

1. (i) Simplify $(2^{2n+1}-2^{2n-1})\div 4^n$.

(ii) Express in factorials $(2n+1)(2n+3)(2n+5)\ldots(4n-1)$, if n is a positive integer.

2. In how many ways can n boys be arranged in a line so that two particular boys have one boy between them ?

3. Find the sum to n terms of the series,

(i) $2(3^2-1^2)+3(4^2-2^2)+4(5^2-3^2)+\ldots$.

(ii) $\dfrac{1}{(n-1)!\,1!}+\dfrac{1}{(n-2)!\,2!}+\dfrac{1}{(n-3)!\,3!}+\ldots$.

4. (i) Write down the first 5 terms of the expansion of $(1+\frac{2}{3}x)^{\frac{3}{2}}$.

(ii) Find the coefficient of x^n in the expansion of $\dfrac{1-2x}{(1-x)(1-3x)}$ in powers of x.

5. Expand in powers of x, up to x^3,

(i) $e^{3x}-2e^x+e^{-x}$; (ii) $\log\dfrac{1+2x}{1-2x}-\log\left(\dfrac{1+x}{1-x}\right)^2$.

A. 13

1. Form the equation whose roots are the squares of the roots of $x^2-2x=5$.

2. Express in partial fractions $\dfrac{x-2}{(x+1)^2(x-1)}$.

Find the coefficient of x^n, if this function can be expanded in ascending powers of x.

3. Examine the convergence of the series whose nth terms are

(i) $\dfrac{1+n^2}{1+n^3}$; (ii) $(-1)^n\dfrac{1+n^2}{1+n^3}$.

4. Sum the series

(i) $1+(1+2)+(1+2+2^2)+\ldots$ n terms.

(ii) $\dfrac{1}{2!}+\dfrac{1+2}{3!}+\dfrac{1+2+2^2}{4!}+\ldots$ to infinity.

5. (i) If x is small, obtain an approximation for $\sqrt[5]{(1+x)}$ in the form $\dfrac{1+ax}{1+bx}$.

(ii) If $|x|<1$, expand $\log\dfrac{(1+x)^2}{1+x^2}$ in powers of x, giving the coefficient of x^n if n is (i) odd, (ii) even.

A. 14

1. Simplify

(i) $4 \div (3-\sqrt{5})^2$; (ii) $(\log n^6 - \log n^3) \div (\log n^2 - \log n)$.

2. How many numbers greater than 5000 can be formed with the digits 7, 6, 5, 3, 0, if repetitions are not allowed? How many of these numbers are even?

3. (i) Find the greatest term or terms in the expansion of $(1-3x)^{-3}$ when $x = \frac{1}{4}$.

(ii) Find the sum to infinity of

$$\frac{5}{3.6} + \frac{5.7}{3.6.9} + \frac{5.7.9}{3.6.9.12} + \dots .$$

4. Prove by induction, or otherwise, that the sum to n terms of $1^4 - 2^4 + 3^4 - 4^4 + \dots$ is $(-1)^{n-1}\frac{1}{2}n(n^3+2n^2-1)$.

5. (i) If $|x| < 1$, prove that the sum to infinity of

$$\frac{x}{1.2} - \frac{x^2}{2.3} + \frac{x^3}{3.4} - \dots$$

is $$\frac{x+1}{x}\log(1+x) - 1.$$

(ii) Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - \log(e+ex)}{x^2}$.

A. 15

1. (i) Simplify $\{\sqrt{(x+y)} - \sqrt{x}\}\{\sqrt{(x+y)} + \sqrt{x}\}$.

(ii) Evaluate $\sum\limits_{1}^{n} \dfrac{1}{\sqrt{(r+1)} + \sqrt{r}}$.

2. In how many ways can $3n$ people be divided up into three equal groups (i) if there is no restriction, (ii) if three particular people are in different groups?

3. If c_r is the coefficient of x^r in $(1+x)^n$,

prove that $3c_0 + 3^2 \cdot \frac{c_1}{2} + 3^3 \cdot \frac{c_2}{3} + 3^4 \cdot \frac{c_3}{4} + \dots$ to $(n+1)$ terms equals

$$\frac{1}{n+1}(4^{n+1}-1).$$

4. Examine the convergence of the series whose nth terms are

$$\text{(i) } \frac{x^n}{n+1}; \quad \text{(ii) } (-1)^n \cdot \frac{n}{2n+1}.$$

5. (i) Find the coefficient of x^4 in the expansion of $(1+x+x^2)^{\frac{1}{2}}$.

(ii) Prove that $\frac{1}{4} - \frac{1}{2.4^2} + \frac{1}{3.4^3} - \frac{1}{4.4^4} + \dots$ equals

$$\frac{1}{5} + \frac{1}{2.5^2} + \frac{1}{3.5^3} + \frac{1}{4.5^4} + \dots .$$

A. 16

1. If $f(t) \equiv \dfrac{t}{e^t - 1} + \frac{1}{2}t$, prove that $f(t) = f(-t)$.

Find the coefficient of t^4, if $f(t)$ is expanded in powers of t.

2. Express $\dfrac{3x^2 - 2x + 1}{x^3 - 3x^2 + 2x}$ in the form

$$\frac{a}{x-2} + \frac{b}{(x-1)(x-2)} + \frac{c}{x(x-1)(x-2)},$$

where a, b, c are constants.

3. Sum the series

(i) $1.3.5 + 2.4.6 + 3.5.7 + \dots$ to n terms;

(ii) $\dfrac{1}{2.3} + \dfrac{1}{4.5} + \dfrac{1}{6.7} + \dots$ to infinity.

4. Find the coefficient of x^n in the expansions of

(i) $\left\{\dfrac{1}{\sqrt{(1-4x)}}\right\}^3$; (ii) $\dfrac{(4x+3)^n}{(x+1)^2}$.

5. (i) If x is small, prove that

$$e^{3x} - 4e^{x} + 6e^{-x} - 4e^{-3x} + e^{-5x} \simeq 16(x^4 - x^5).$$

(ii) Solve the equation $\log(1+x) = \frac{1}{2}$, correct to 3 places of decimals.

A. 17

1. If α, β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $a\alpha + b$, $a\beta + b$.

2. If $|x| > 1$ and if $\dfrac{1 + 3x^2}{(1+x)(1-x)^2}$ is expanded in powers of $\dfrac{1}{x}$, prove that the coefficient of $\dfrac{1}{x^n}$ is $2n+1$ or $2n-1$ according as n is odd or even.

3. How many permutations can be formed from the letters *aabbcde*, taken 4 at a time?

4. Find the sums to infinity of the series:

(i) $\dfrac{1}{2.3.4} + \dfrac{1}{4.5.6} + \dfrac{1}{6.7.8} + \dots$.

(ii) $1 + \dfrac{2^2}{2!} + \dfrac{3^2}{3!} + \dfrac{4^2}{4!} + \dots$.

5. (i) If x is large, prove that

$$1 - \sqrt{\left(\frac{x-1}{x+1}\right)} \simeq \frac{1}{x} - \frac{x-1}{2x^3}.$$

(ii) Evaluate $\displaystyle\lim_{x \to 0} \frac{(2+x)\log(1+x) + (2-x)\log(1-x)}{x^4}$.

A. 18

1. Express 1933 in the form

$$a_1 + a_2(2!) + a_3(3!) + a_4(4!) + \dots$$

where, for all integral values of r, $a_r < r + 1$.

2. (i) Find the number of integers which are factors of 8400, excluding 1 and 8400.

(ii) Find the number of terms in the expansion of

$$(a + b + c + d)^7.$$

3. If $\dfrac{(x-4)(x-5)}{(x-2)(x-3)}$ can be expanded in ascending powers of x, find the coefficient of x^n.

4. Find the sums to infinity of the series,

(i) $1 + \dfrac{3^2}{1!} + \dfrac{5^2}{3!} + \dfrac{7^2}{5!} + \dots$;

(ii) $\dfrac{1}{1.2.3} - \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} - \dfrac{1}{4.5.6} + \dots$.

5. (i) If x is large, prove that

$$\sqrt{(x^2 + x)}\{\log(x+1) - \log x\} \simeq 1 - \frac{1}{24x^2} + \frac{1}{24x^3}.$$

(ii) Prove that $\lim\limits_{x \to 0} \dfrac{1}{x^3}(e^{2x} - 3e^x + 3 - e^{-x}) = 1$.

A. 19

1. Find the condition that both roots of $ax^2 + bx + c = 0$ are (i) positive, (ii) greater than 2.

2. In how many ways can a total of 30 marks be assigned to 5 questions, if no question receives less than 3 marks or more than 8 marks?

3. The expression $a + \dfrac{b}{1+x} + \dfrac{c+dx}{1+x+x^2}$ can be expanded in the form, $1 + x + x^2 + x^3 + px^4 + qx^5 + \dots$, when $|x| < 1$; find the values of a, b, c, d, p, q.

4. Examine the convergence of the series whose nth terms are

$$\text{(i) } \frac{x^n}{\sqrt{(1+n^2)}}; \quad \text{(ii) } \frac{e^{nx}}{n!}.$$

5. (i) Prove that $e^n > \dfrac{(1+n)^n}{n!}$, if n is a positive integer.

(ii) Show that the coefficient of x^n in the expansion of

$$\log(1 + 2x + 2x^2 + x^3)$$

has one of the values 0, $-\dfrac{1}{n}$, $\dfrac{2}{n}$, $-\dfrac{3}{n}$, and distinguish the cases.

A. 20

1. If the equations $ax+by=1$, $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ have only one solution in x and y, prove that $a^2c^2+b^2d^2=1$ and find the solution.

2. If n straight lines are drawn in a plane so that no two of them are parallel and no three of them pass through the same point, the number of regions into which the plane is divided is denoted by $f(n)$. What are the values of $f(1)$, $f(2)$, $f(3)$? Prove that $f(n+1)-f(n)=n+1$ and deduce that $f(n)=\frac{1}{2}(n^2+n+2)$.

3. If $\frac{b}{a}$ is so small that $\frac{b^3}{a^3}$ is negligible, prove that the sum to n terms of the H.P., $\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\ldots$, is

$$\frac{n}{a}-\frac{1}{2}\frac{n(n+1)b}{a^3}\{a-\tfrac{1}{3}(2n+1)b\}.$$

4. (i) If p is small, prove that a root of $xe^x=p$ is $p-p^2$ approximately.

(ii) Expand $\log\frac{1+x+x^2}{1-x+x^2}$ in powers of x if $|x|<1$.

5. It can be proved that the sum to infinity of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ is $\frac{\pi^2}{6}$. Assuming this fact, prove that the sum to infinity of the series $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ is $\frac{\pi^2}{8}$ and that the sum to infinity of $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is $\frac{\pi^2}{12}$.

CHAPTER VII

RATIONAL FUNCTIONS

THE two previous chapters have shown that the methods of the Calculus are often of great assistance in studying the behaviour of a function. In this chapter, we shall use the following property:

If $f(x)$ is a function which possesses a derivative $\dfrac{d}{dx}f(x)$ for all values of x in $a \leqslant x \leqslant b$, and if $\dfrac{d}{dx}f(x) > 0$ whenever $a < x < b$, then $f(x)$ steadily increases when x increases from a to b; also if $\dfrac{d}{dx}f(x) < 0$ whenever $a < x < b$, then $f(x)$ steadily decreases when x increases from a to b.

We shall also make use of the tests for maximum and minimum values of $f(x)$ which follow from this property.

The derivative $\dfrac{d}{dx}f(x)$ is usually denoted by $f'(x)$.

The Quadratic Function

The properties of the function,

$$px^2 + 2qx + r, \quad p \neq 0,$$

have been discussed in *New Algebra*, Part III, see p. 133, by using the identity,

$$px^2 + 2qx + r \equiv p\left\{\left(x + \frac{q}{p}\right)^2 - \frac{q^2 - pr}{p^2}\right\}.$$

Suppose $f(x) \equiv \left(x + \dfrac{q}{p}\right)^2 - \dfrac{q^2 - pr}{p^2}$, $p \neq 0$. $\left(x + \dfrac{q}{p}\right)^2$ takes values as large as we please for all sufficiently large values of x and for all sufficiently numerically large negative values of x; therefore $f(x) \to \infty$ when $x \to \infty$ and when $x \to -\infty$.

This is often expressed symbolically in the form,

$$f(\infty) = \infty \quad \text{and} \quad f(-\infty) = \infty\ ;$$

if, however, this notation is employed, it must be clearly understood that it is merely a short-hand method of stating the previous sentence.

Further, since $f(x) \equiv \left(x+\frac{q}{p}\right)^2 - \frac{q^2-pr}{p^2}$, when x increases from $-\infty$ to $-\frac{q}{p}$, $f(x)$ steadily decreases from ∞ to $-\frac{q^2-pr}{p^2}$, and when x increases from $-\frac{q}{p}$ to ∞, $f(x)$ steadily increases from $-\frac{q^2-pr}{p^2}$ to ∞.

If $q^2 < pr$, $f(x)$ is always positive, and $f(x)=0$ has no (real) root.

If $q^2 > pr$, $f(x)=0$ has two (real) roots, $x = \frac{-q \pm \sqrt{(q^2-pr)}}{p}$.

If $q^2 = pr$, $f(x) \equiv \left(x+\frac{q}{p}\right)^2 = 0$ is said to have two equal roots or a "repeated root," $x = -\frac{q}{p}$.

[For a note on the use of the word "real," see *New Algebra*, Part III, p. 130. The word will not be used in this book, except in some of the exercises, where the old-fashioned form of words is employed to conform with examination practice.]

The behaviour of $f(x)$ may be examined by using Calculus methods, and for more complicated functions this is often the best procedure.

Thus, since $f(x) \equiv x^2 + \frac{2qx}{p} + \frac{r}{p}$, $f'(x) = 2x + \frac{2q}{p}$.

$$\therefore\ f'(x) < 0 \text{ if } x < -\frac{q}{p} \quad \text{and} \quad f'(x) > 0 \text{ if } x > -\frac{q}{p}.$$

$\therefore$ as x increases from $-\infty$ to $-\frac{q}{p}$, $f(x)$ steadily decreases, and as x increases from $-\frac{q}{p}$ to ∞, $f(x)$ steadily increases; and the least value of $f(x)$ is given by $x = -\frac{q}{p}$.

EXERCISE VII. a

Find for what values of x the following functions increase as x increases:

1. $x^2 - 10x - 7$. **2.** $3x^2 + 4x - 5$. **3.** $8 - 6x - 3x^2$.

Find the maximum or minimum values of the following functions:

4. $x^2 + 6x - 10$. **5.** $3 - x - x^2$. **6.** $4x^2 + 4x + 1$.

7. What is the least value of $2x^2 - x$? What is the condition that $2x^2 - x = c$ has (i) two (real) roots, (ii) a repeated root?

8. What is the greatest value of $(x-2)(5-x)$? What is the condition that $(x-2)(5-x) = c$ has (i) no (real) roots, (ii) a repeated root?

Sketch graphs of the following functions, without making a table of values:

9. $x^2 - 7x + 12$. 10. $(x+1)(7-x)$. 11. $x^2 - 6x + 10$.

12. If $px^2 + qx + r = 0$ has a repeated root, find its value in terms of (i) p, q; (ii) q, r.

13. Prove that $4x^2 + 8x - 8 = m(4x-3)$ has no (real) roots if $3 < m < 4$.

14. Prove that the roots of

$$(a^2+b^2)x^2 + 2(a^2+b^2+c^2)x + b^2 + c^2 = 0,$$

are real, if a, b, c are real.

15. Prove that the roots of

$$(b^2 - 4ac)x^2 + 4(a+c)x = 4$$

are real if a, b, c are real, and find the condition that they are equal.

16. If $a > b > 0$, prove that $x(x-a) = k(x-b)$ has real roots, whatever real value k may have.

17. Find the condition that the equations

$$ax + by = x^2 + xy + y^2 = 1$$

may have real solutions, if a and b are real.

18. Find the range of values of λ for which the two equations, $x^2 + 4x + \lambda = 5$, $(x-2)^2 - \lambda = 4$, are such that one root of each equation lies in value between the two roots of the other.

19. Find the values of m for which the expression,

$$2(x^2 - 3x + 4) - m(x^2 - x - 2)$$

is a perfect square. Hence find constants such that

$$x^2 - 3x + 4 \equiv a(x-p)^2 + b(x-q)^2$$

and

$$x^2 - x - 2 \equiv c(x-p)^2 + d(x-q)^2.$$

The Cubic Function

Example 1. Sketch the graph of

$$f(x) \equiv (x+3)(x-2)(x-5)$$

and find its turning values.

Since $f(x) = 0$ if $x = -3$ or 2 or 5, the graph crosses the x-axis at -3, 2, 5.

Also $f(x)\to\infty$ when $x\to\infty$ and $f(x)\to-\infty$ when $x\to-\infty$; therefore, using the convention of p. 136, the behaviour of $f(x)$ may be tabulated as follows:

x	$-\infty$	$-\infty$ to -3	-3	-3 to 2	2	2 to 5	5	5 to ∞	∞
$f(x)$	$-\infty$	$-$	0	$+$	0	$-$	0	$+$	∞

Since $f(x)$ is continuous for all values of x, the *form* of the graph of $y=f(x)$ can now be drawn, but a knowledge of the turning points makes it possible to draw it more accurately.

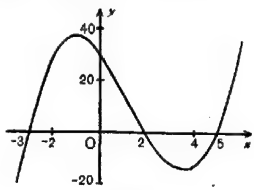

FIG. 6.

We have $f(x)\equiv x^3-4x^2-11x+30$,

$\therefore\ f'(x)\equiv 3x^2-8x-11=(x+1)(3x-11)$.

$\therefore\ f'(x)>0$ if $x<-1$ and $f'(x)<0$ if $-1<x<\frac{11}{3}$.

$\therefore\ x=-1$ gives a maximum value of $f(x)$, and this value is $f(-1)=36$.

Similarly $x=\frac{11}{3}$ gives the minimum value $f(\frac{11}{3})=-14\frac{22}{27}$.

Example 2. Discuss the function, $f(x)\equiv x^3-9x^2+15x-2$.

It is easy to see that for sufficiently large values of x, $f(x)$ takes values as large as we please; thus if $x>18$, $9x^2<\frac{1}{2}x^3$ and $15x>2$, therefore $f(x)>\frac{1}{2}x^3$.

$$\therefore\ f(x)\to\infty \text{ when } x\to\infty.$$

It is also evident that for sufficiently numerically large negative values of x, $f(x)$ takes negative values as numerically large as we please; $\therefore\ f(x)\to-\infty$ when $x\to-\infty$.

Using the convention explained on p. 136, we write

$$f(\infty)=\infty,\quad f(-\infty)=-\infty.$$

Since $f(x)$ is continuous for all values of x, its graph must cross the x-axis at one point at least;

$$\therefore\ f(x)=0 \text{ has at least one root.}$$

Further, $\quad f'(x)=3x^2-18x+15=3(x-1)(x-5)$.

$\therefore\ f'(x)=0$ when $x=1$ and when $x=5$, and $f(1)=5, f(5)=-27$.

Also $f'(x)>0$ if $x<1$ and if $x>5$; $f'(x)<0$ if $1<x<5$;

$\therefore$ when x increases from $-\infty$ to 1, $f(x)$ steadily increases from $-\infty$ to 5.

When x increases from 1 to 5, $f(x)$ steadily decreases from 5 to -27.

When x increases from 5 to ∞, $f(x)$ steadily increases from -27 to ∞.

But $f(x)$ is continuous for all values of x; the graph of $y=f(x)$ therefore takes the form shown in the figure. It crosses the x-axis at three points, firstly to the left of $x=1$, secondly between $x=1$ and $x=5$, thirdly to the right of $x=5$.

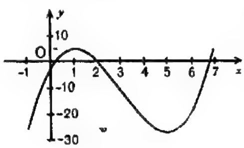

FIG. 7.

$$\therefore\ x^3-9x^2+15x-2=0$$

has 3 roots.

The positions of these roots can be determined more precisely by evaluating $f(x)$ for additional values of x. Thus we have $f(0)=-2, f(1)=5, f(5)=-27, f(7)=5$;

$\therefore$ the roots lie respectively in the intervals $0<x<1$, $1<x<5$, $5<x<7$.

The reader can easily check the conclusions of this example algebraically because the function equals $(x-2)(x^2-7x+1)$.

It is instructive to compare this graph with those of the functions (see Ex. VII b, No. 13),

$$\text{(i) } x^3-9x^2+15x+25\,; \quad \text{(ii) } x^3-9x^2+15x+74\,;$$

which are obtained by moving this graph parallel to the y-axis through distances of 27 units and 76 units respectively.

The function $f'(x)$ is unaltered, and the turning points are given by the same values of x. But the graph of (i) now touches the x-axis at $x=5$, so that $\quad x^3-9x^2+15x+25=0$

has a repeated root $x=5$ and one other root.

And the "minimum" value of (ii) is positive, so that

$$x^3 - 9x^2 + 15x + 74 = 0$$

has only one root.

These results may be checked algebraically by noticing that (i) equals $(x+1)(x-5)^2$ and (ii) equals $(x+2)(x^2-11x+37)$.

One of the best ways of investigating the behaviour of a function $f(x)$ is to examine the nature of the roots of the equation $f'(x)=0$. In Examples 1, 2, a function $f(x)$ was chosen such that $f'(x)=0$ had two roots. For comparison, we shall next discuss a function $f(x)$ where $f'(x)=0$ has no roots.

Example 3. Discuss the function, $f(x) \equiv x^3 + 3x + 14$.

As in Example 2, we find that $f(x) \to \infty$ when $x \to \infty$, and $f(x) \to -\infty$ when $x \to -\infty$; therefore, since $f(x)$ is continuous for all values of x, its graph must cross the x-axis at one point at least.

Further, $\qquad f'(x) = 3x^2 + 3 = 3(x^2+1)$.

$\therefore$ $f'(x)=0$ has no roots, and $f'(x) > 0$ for all values of x.

$\therefore$ when x increases from $-\infty$ to $+\infty$, $f(x)$ steadily increases from $-\infty$ to $+\infty$.

$\therefore$ there is one, and only one, value of x for which $f(x)=0$; this value is negative since $f(0)=14$ is positive. The graph of $y=f(x)$ therefore takes the form shown in the figure.

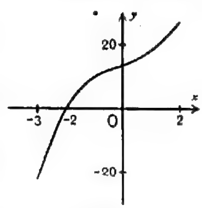

FIG. 8.

The reader can easily check these conclusions algebraically, because the function equals $(x+2)(x^2-2x+7)$.

It may also be noted that since $f'(x) \equiv 3(x^2+1)$ is least when $x=0$, the *slope* of the curve steadily decreases from ∞ to 3 when x increases from $-\infty$ to 0, and then steadily increases from 3 to ∞ when x increases from 0 to ∞; consequently there is a twist in the curve at $x=0$, and this is called a "point of inflexion."

The methods of Examples 2, 3 may be applied to other polynomials.

Suppose $f(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n$.

Then we can prove that $f(x) \to \infty$ when $x \to \infty$ as follows:

If $x > 1$, $x^{n-1} > x^{n-2} > x^{n-3} > \dots$;

$\therefore$ if k is the greatest of the constants, $|p_1|, |p_2|, \dots, |p_n|$,

$$f(x) > x^n - nkx^{n-1} \text{ if } x > 1;$$

$$\therefore f(x) > \tfrac{1}{2}x^n \text{ whenever } x > 2nk + 1.$$

$\therefore$ $f(x)$ takes values as large as we please for all sufficiently large values of x.

Similarly, we can prove that

$$f(x) \to \infty \text{ or } \to -\infty \text{ when } x \to -\infty,$$

according as n is even or odd.

$\therefore$ if n is odd, $f(x) = 0$ has at least one root.

If n is even, $f(x) = 0$ may have no roots; but since $f(0) = p_n$, $f(x) = 0$ has at least one negative root and at least one positive root, if p_n is negative.

Repeated Roots

If $f(x) \equiv (x - a)^2 (x^{n-2} + q_1 x^{n-3} + \dots + q_{n-2})$, we say that $f(x) = 0$ has two equal roots a, or that $x = a$ is a repeated root of $f(x) = 0$.

Since

$$f'(x) = (x - a)^2\{(n-2)x^{n-3} + \dots\} + 2(x - a)\{x^{n-2} + q_1 x^{n-3} + \dots\},$$

it follows that $x - a$ is a factor of $f'(x)$.

And more generally, if $(x - a)^r$ is a factor of the polynomial $f(x)$, then $(x - a)^{r-1}$ is a factor of $f'(x)$.

$\therefore$ repeated roots of $f(x) = 0$ can be obtained by finding the H.C.F. of $f(x)$ and $f'(x)$, see Example 6, p. 143.

These examples illustrate two important general properties, which may be stated, although the proofs are outside the scope of this book.

(I.) If a function $f(x)$ is continuous for the range of values $a \leqslant x \leqslant b$, and if $f(a)$, $f(b)$ are of opposite signs, then the equation $f(x) = 0$ has at least one root between $x = a$ and $x = b$.

Thus, if $f(x) \equiv x^3 - 9x^2 + 15x - 2$, see Example 2, p. 139, $f(0) = -2$ and $f(1) = 5$, so that the points on the curve corresponding to $x = 0$ and $x = 1$ lie on opposite sides of the x-axis. Therefore the graph, since $f(x)$ is continuous, must cut the x-axis somewhere between $x = 0$ and $x = 1$.

(II.) If $f(x)=0$ when $x=a$ and when $x=b$ and if $f'(x)$ exists for all values of x in the range $a<x<b$, then $f'(x)=0$ has at least one root between $x=a$ and $x=b$.

Graphically this means that if the curve crosses the x-axis at two points A and B, and if there exists an ordinary tangent at every point of the arc AB, then at least one of these tangents must be parallel to the x-axis.

Thus, in Example 1, p. 138, the curve crosses the x-axis at three points, $x=-3$, $x=2$, $x=5$, and has a tangent parallel to the x-axis at $x=-1$ between $x=-3$ and $x=2$, and at $x=\frac{11}{3}$ between $x=2$ and $x=5$.

Another way of expressing this property is to say that *all the roots of $f(x)=0$ are separated by roots of $f'(x)=0$, provided that $f'(x)$ exists for the range of values of x considered.*

This fact often helps to locate the roots of $f(x)=0$.

Example 4. Prove that $x^4+3x^3-15x^2-19x+13=0$ has 4 roots.

If $f(x)\equiv x^4+3x^3-15x^2-19x+13$,

$f(-\infty)=+\infty$, $f(-2)=-17$, $f(0)=13$, $f(1)=-17$, $f(\infty)=\infty$.

$\therefore$ $f(x)=0$ has one root <-2, one root between -2 and 0, one root between 0 and 1, and one root >1.

Example 5. Prove that $x^4+4x+4=0$ has no roots.

$$\text{If } f(x)\equiv x^4+4x+4,\ f'(x)=4x^3+4=4(x^3+1).$$

$$\therefore\ f'(x)<0 \text{ if } x<-1, \text{ and } f'(x)>0 \text{ if } x>-1.$$

$\therefore$ the least value of $f(x)$ is $f(-1)=1$, and since this is positive, there is no value of x for which $f(x)=0$.

$$\therefore\ x^4+4x+4=0 \text{ has no roots.}$$

Example 6. Solve $12x^3-28x^2+3x+18=0$, given that it has two equal roots.

$$\text{If } f(x)=12x^3-28x^2+3x+18,$$

$$f'(x)=36x^2-56x+3=(2x-3)(18x-1).$$

Since $f(x)=0$ has a repeated root, either $(2x-3)^2$ or $(18x-1)^2$ is a factor of $f(x)$.

By trial, $f(x)=(4x^2-12x+9)(3x+2)$;

$$\therefore \text{ the roots are } \tfrac{3}{2},\ \tfrac{3}{2},\ -\tfrac{2}{3}.$$

EXERCISE VII. b

1. Show that the roots of $x^2 - 11x + 9 = 0$ lie between 0 and 1, 10 and 11.

2. Show that the roots of $\dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{1}{x-3} = 0$ lie between 1 and 2, 2 and 3.

Find for what values of x the following functions increase as x increases. What values of x give turning values of the function? State whether the value is a maximum or minimum.

3. $x^3 - 12x$. 4. $2 + 3x - x^3$. 5. $x^3 + x - 2$.

6. $x^3 - 6x^2 + 9x - 4$. 7. $x^4 - 8x^2 + 12$. 8. $x^4 - 4x^3 - 20x^2 - 49$.

9. Discuss the function $4x^3 - 12x + c$ if (i) $c = 2$, (ii) $c = 8$, (iii) $c = 10$. Sketch graphs to illustrate your work.

10. Discuss the function $x^3 + 12x - 20$ and sketch its graph.

11. Sketch the graph of $x(x-5)(x-8)$, showing the positions of the turning points.

12. Repeat No. 11 for the function $(x-1)(x-5)(2x-7)$.

13. Sketch on the same figure the graphs of $x^3 - 9x^2 + 15x + c$, if (i) $c = 0$, (ii) $c = 25$, (iii) $c = 74$; and find the nature of the roots of $x^3 - 9x^2 + 15x + c = 0$ in these three cases.

14. (i) Show that the roots of $x^3 - 3x^2 - 4x + 4 = 0$ lie between -2 and -1, 0 and 1, 3 and 4.

(ii) Show that $x^3 - 3x^2 - 4x + 19 = 0$ has a root between -3 and -2 and no other roots.

15. Prove that $x^3 - 2x^2 = 1$ has a root between 2 and 3 and no other roots.

16. Prove that $2x^3 + x^2 = 1$ has a root between 0 and 1 and no other roots.

17. Prove that $x^3 - 4x = 1$ has 3 (real) roots.

18. Prove that $x^3 - 12x^2 + 45x = 51$ has 3 (real) roots.

19. Prove that $(x-1)(x-3)(x-5) + k(x-2)(x-4)(x-6) = 0$ has 3 (real) roots if $k \neq -1$, and if k is positive find roughly their positions.

20. Prove that $2x^3 - 7x^2 - 12x + 45 = 0$ has a repeated root and find it.

21. Solve $4x^4 - 20x^3 + 13x^2 = 1$, given that there are two equal roots.

22. Prove that $4x^4 - 2x + c = 0$ has no roots if $c > \frac{3}{4}$. What can you say about the roots if $c = \frac{3}{4}$? Prove that the equation cannot have more than 2 roots, whatever value c has.

23. If $a > b > c > 0$, prove that

$$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=\frac{1}{x}$$

has 3 roots, of which one is negative, and the others lie between a and b, b and c.

24. Find the condition that $x^3 - 3p^2x + q = 0$ has a repeated root.

25. If $x^3 - 3qx + r = 0$ has 3 (real) roots, prove that q must be positive. What are the turning values of $x^3 - 3qx + r$? Hence find the sufficient condition for 3 (real) roots.

The Function $\dfrac{ax^2+2bx+c}{Ax^2+2Bx+C}$

In the previous examples, we have repeatedly made use of the fact that polynomials, and their derivatives, are continuous functions for all values of the variable. Rational functions, however, may be discontinuous for special values of the variable, as was shown in *New Algebra*, Part III, pp. 97-100.

Example 7. Discuss the function,

$$f(x) \equiv \frac{(2x-1)(x-8)}{(x-1)(x-4)}.$$

$f(x) = 0$ when $x = \frac{1}{2}$, $x = 8$, and has no value when $x = 1$, $x = 4$. The function is therefore continuous only *throughout* such ranges as exclude the values $x = 1$, $x = 4$.

It is convenient to use the notation $x = 1-$ to denote values of x near 1 less than 1, and $x = 1+$ to denote values of x near 1 greater than 1.

Thus for $x = 1-$, $f(x)$ is numerically large and negative, which for brevity will be called "large $-$," and for $x = 1+$, $f(x)$ is large and positive, which will be called "large $+$."

Put $\dfrac{(2x-1)(x-8)}{(x-1)(x-4)} = y$, then $y = \dfrac{2x^2-17x+8}{x^2-5x+4} = 2 - \dfrac{7x}{x^2-5x+4}$;

$\therefore$ if x is large $+$ or large $-$, $y \doteq 2 - \dfrac{7}{x}$.

$\therefore$ if x is large $-$, $y = 2+$, and if x is large $+$, $y = 2-$.
Also $y = 2$ when $x = 0$ and for no other value of x.

The behaviour of $y=f(x)$ may therefore be tabulated as follows :

x	$-\infty$	$-\infty$ to 0	0	0 to $\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ to $1-$	$1-$	1
$f(x)$	$2+$	$+$	2	$+$	0	$-$	large $-$	no value

x	$1+$	$1+$ to $4-$	$4-$	4
$f(x)$	large $+$	$+$	large $+$	no value

x	$4+$	$4+$ to 8	8	8 to ∞	∞
$f(x)$	large $-$	$-$	0	$+$	$2-$

The reader should now make a sketch of the graph which represents the facts in this table.

These facts are sufficient for a rough diagram, but a clearer idea of the behaviour of the function will be obtained by finding the turning points of the graph.

We have $$y=2-\frac{7x}{x^2-5x+4}.$$

Put $-\dfrac{7x}{x^2-5x+4}=z$ so that $y=2+z$.

Then $zx^2-x(5z-7)+4z=0$.

This equation in x has no roots if $(5z-7)^2-16z^2<0$, that is, if $(9z-7)(z-7)<0$.

$\therefore$ there are no values of x for which $\frac{7}{9}<z<7$.

$\therefore$ y has no value between $2\frac{7}{9}$ and 9, and we find easily that $y=2\frac{7}{9}$ when $x=-2$ and that $y=9$ when $x=2$.

The reader should now modify his sketch by using these facts which, when considered in conjunction with his former sketch, suggest that there is a "maximum" value $2\frac{7}{9}$ at $x=-2$ and a "minimum" value 9 at $x=2$.

The accuracy of the form of the graph can be tested by calculating $f'(x)$.

Since $f(x)=2-\dfrac{7x}{x^2-5x+4}$,

$$f'(x)=-\frac{(x^2-5x+4)7-7x(2x-5)}{(x^2-5x+4)^2}=\frac{7(x+2)(x-2)}{(x-1)^2(x-4)^2};$$

$\therefore$ $f'(x)=0$ when $x=-2$ and when $x=2$, and $f(-2)=2\frac{7}{9}$, $f(2)=9$.

Also $f'(x) > 0$ if $x < -2$ and if $2 < x < 4$ and if $x > 4$;

$f'(x) < 0$ if $-2 < x < 1$ and if $1 < x < 2$.

$\therefore$ when x increases from $-\infty$ to -2, $f(x)$ steadily increases from 2 to $2\frac{7}{9}$; when x increases from -2 to 1, $f(x)$ steadily decreases from $2\frac{7}{9}$ to $-\infty$.

At $x=1$, there is a break in the curve.

When x increases from 1 to 2, $f(x)$ steadily decreases from ∞ to 9;

when x increases from 2 to 4, $f(x)$ steadily increases from 9 to ∞.

At $x=4$, there is a break in the curve.

When x increases from 4 to ∞, $f(x)$ steadily increases from $-\infty$ to 2.

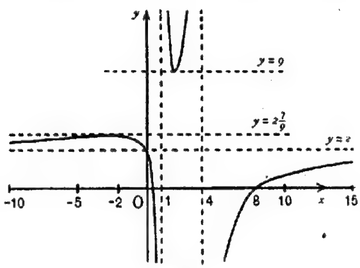

FIG. 9.

The graph of $y=f(x)$ therefore takes the form shown in the figure, but owing to considerations of space, part of it is out of scale.

This graph illustrates the difference of meaning of the phrases "maximum value" and "greatest value," and between "minimum value" and "least value." $f(x)$ is said to have a maximum value $2\frac{7}{9}$ at $x = -2$, because its value there is greater than that at all other points in the *neighbourhood* of $x = -2$; a maximum corresponds to the top of a hill. Similarly $f(x)$ has a minimum value 9 at $x=2$, because its value there is less than that at all other points in the neighbourhood of $x=2$.

For this function, the maximum value happens to be less than the minimum value, but this fact is irrelevant. It should also be noted that actually this function has no greatest value, and no

least value. Whatever positive number K is named, however large, values of x (near 1 or near 4) can be found such that $f(x)$ is greater than K, and values can also be found such that $f(x) < -K$.

Example 8. Discuss the function, $f(x) \equiv \dfrac{(x+2)(x-3)}{(x+1)(x-6)}$.

$f(x)=0$ when $x=-2$, $x=3$, and has no value when $x=-1$, $x=6$. The function is therefore continuous only *throughout* such ranges as exclude $x=-1$, $x=6$. Also for $x=6-$ and for $x=-1-$, $f(x)$ is large $-$; and for $x=6+$ and for $x=-1+$, $f(x)$ is large $+$.

Put $\dfrac{(x+2)(x-3)}{(x+1)(x-6)}=y$, $\quad\therefore\ y=\dfrac{x^2-x-6}{x^2-5x-6}=1+\dfrac{4x}{x^2-5x-6}$;

$\therefore$ if x is large $+$ or large $-$, $y \simeq 1+\dfrac{4}{x}$;

$\therefore$ if x is large $-$, $y=1-$, and if x is large $+$, $y=1+$.

Also $y=1$ when $x=0$ and for no other value of x.

The behaviour of $y=f(x)$ may therefore be tabulated as follows:

x	$-\infty$	$-\infty$ to -2	-2	-2 to $-1-$	$-1-$	-1	$-1+$
$f(x)$	$1-$	$+$	0	$-$	large $-$	no value	large $+$

x	$-1+$ to 3	3	3 to $6-$	$6-$	6	$6+$	$6+$ to ∞	∞
$f(x)$	$+$	0	$-$	large $-$	no value	large $+$	$+$	$1+$

The reader should now sketch a graph which represents the facts in this table.

The accuracy of the form of the graph can be tested by calculating $f'(x)$.

Thus $f'(x)=\dfrac{(x^2-5x-6)4-(4x)(2x-5)}{(x^2-5x-6)^2}=\dfrac{-4(x^2+6)}{(x+1)^2(x-6)^2}$.

Since there is no value of x for which $f'(x)$ is zero, the graph has no turning points; and since $f'(x)$ is negative for all values of x for which it exists, $f(x)$ steadily decreases throughout each range of values of x *in which $f(x)$ is continuous.*

$\therefore$ $f(x)$ steadily decreases as x increases from $-\infty$ to -1, and does so again as x increases from -1 to 6, and again as x increases from 6 to ∞.

The graph of $y=f(x)$ therefore takes the form shown in the figure.

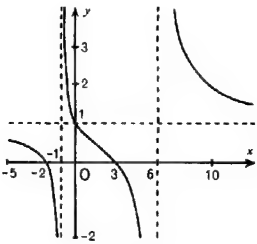

FIG. 10.

Example 9. Discuss the function, $f(x)\equiv\dfrac{(x-1)(x-3)}{5x^2+4}$.

$f(x)=0$ when $x=1$, $x=3$, and since there is no value of x for which $5x^2+4$ is zero, $f(x)$ is continuous for all values of x.

Put $\dfrac{(x-1)(x-3)}{5x^2+4}=y$, then $y=\dfrac{x^2-4x+3}{5x^2+4}=\dfrac{1}{5}-\dfrac{20x-11}{5(5x^2+4)}$;

$$\therefore \text{ if } x \text{ is large} + \text{ or large} -,\ y \simeq \frac{1}{5}-\frac{4}{5x}.$$

$\therefore$ if x is large $-$, $y=\frac{1}{5}+$, and if x is large $+$, $y=\frac{1}{5}-$.
Also $y=\frac{1}{5}$ when $x=\frac{11}{20}$ and for no other value of x.

The behaviour of $y=f(x)$ may therefore be tabulated as follows:

x	$-\infty$	$-\infty$ to $\frac{11}{20}$	$\frac{11}{20}$	$\frac{11}{20}$ to 1	1	1 to 3	3	3 to ∞	∞
$f(x)$	$\frac{1}{5}+$	$+$	$\frac{1}{5}$	$+$	0	$-$	0	$+$	$\frac{1}{5}-$

The reader should now make a sketch of the graph which represents the facts in this table; it will then be evident that greater accuracy will be secured if the turning points are known.

Since $y=\dfrac{x^2-4x+3}{5x^2+4}$, we have

$$x^2(1-5y)-4x+3-4y=0.$$

This equation in x has no roots if $4-(1-5y)(3-4y)<0$, that is, if $1+19y-20y^2\equiv(1+20y)(1-y)<0$.

$\therefore$ there are no values of x for which $y < -\frac{1}{20}$ or for which $y > 1$, and we find easily that $y = -\frac{1}{20}$ when $x = 1\frac{3}{5}$ and that $y = 1$ when $x = -\frac{1}{2}$.

The reader should now modify his sketch by using these facts which, when considered in conjunction with his former sketch, suggest that there is a maximum value 1 at $x = -\frac{1}{2}$, and a minimum value $-\frac{1}{20}$ when $x = 1\frac{3}{5}$.

As before, the work may be tested by calculating $f'(x)$.

$$f'(x) = \frac{(5x^2+4)(2x-4)-(x^2-4x+3)\,10x}{(5x^2+4)^2} = \frac{2(2x+1)(5x-8)}{(5x^2+4)^2}.$$

$\therefore f'(x) = 0$ when $x = -\frac{1}{2}$ and when $x = 1\frac{3}{5}$.

Also $f'(x) > 0$ if $x < -\frac{1}{2}$ and if $x > 1\frac{3}{5}$;

$f'(x) < 0$ if $-\frac{1}{2} < x < 1\frac{3}{5}$.

Therefore, as in Example 7, $x = -\frac{1}{2}$ gives a maximum and $x = 1\frac{3}{5}$ gives a minimum.

The graph of $y = f(x)$ therefore takes the form shown in the figure.

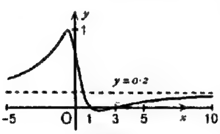

FIG. 11.

Examples 7-9 illustrate the principal varieties of the forms which the graph of $f(x) \equiv \dfrac{ax^2+2bx+c}{Ax^2+2Bx+C}$ may take. The conclusions may be summarised as follows:

(i) If $B^2 > AC$, $Ax^2+2Bx+C=0$ has two unequal roots, say $x=\alpha$, $x=\beta$; and therefore there are breaks in the graph of $f(x)$ at $x=\alpha$ and at $x=\beta$.

Further, if $f'(x)=0$ has two roots, as in Example 7, the function has a maximum and a minimum value.

But if $f'(x)=0$ has no roots, as in Example 8, the function steadily increases throughout each of the intervals $-\infty$ to α, α to β, β to $-\infty$, where $\alpha < \beta$, *or* steadily decreases throughout each of these intervals.

It may also happen that $f'(x)=0$ has only one root, and this will be so if $f(x)$ can be reduced to the form $\dfrac{a}{A} + \dfrac{k}{Ax^2+2Bx+C}$, in which case $f(x)$ has either a maximum or a minimum value, but not both.

(ii) If $B^2 < AC$, $Ax^2 + 2Bx + C = 0$ has no roots, and therefore $f(x)$ is continuous for all values of x. In this case, it will be found that $f'(x) = 0$ always has roots (or at least one root), and the function has, as in Example 9, a *greatest* and a *least* value (or limiting value).

(iii) If $B^2 = AC$, $Ax^2 + 2Bx + C$ takes the form $A(x - a)^2$ and therefore there is *one* break, $x = a$, in the graph, and it will be found that $f(x)$ has either a *greatest* or a *least* value (or limiting value), but not both.

Example 10. Prove that $\dfrac{x^2 + 2x + c}{x^2 + 4x + 3c}$ can assume any value if $0 < c < 1$.

If $\dfrac{x^2 + 2x + c}{x^2 + 4x + 3c} = y$, we have

$$x^2(1 - y) + 2x(1 - 2y) + c(1 - 3y) = 0.$$

This equation has roots if

$$(1 - 2y)^2 - c(1 - y)(1 - 3y) > 0,$$

that is, if $\quad (1 - 2y)^2(1 - c) + cy^2 > 0,$

and this is true for all values of y if $0 < c < 1$.

EXERCISE VII. c

Discuss the following functions. Find their turning values, if any, and sketch graphs to illustrate your work.

1. $\dfrac{x}{(x - 1)(x - 4)}$. 2. $\dfrac{x}{x^2 + 1}$. 3. $\dfrac{x}{x^2 - 1}$.

4. $\dfrac{x + 1}{(x - 1)^2}$. 5. $\dfrac{1}{(x - 1)(x - 5)}$. 6. $\dfrac{x^2}{x^2 + 4}$.

7. $\dfrac{(2x + 3)(x - 6)}{(x + 1)(x - 2)}$. 8. $\dfrac{(3x - 1)(2x + 1)}{(6x - 1)(x + 1)}$. 9. $\dfrac{(x + 2)(x - 3)}{x^2 + x + 1}$.

10. For what ranges of values of x is $\dfrac{x^2 + 3x - 4}{x^2 - x - 2}$ positive? Sketch the graph of this function.

11. Find the limits between which the values of $\dfrac{x^2 + 2x + 3}{x^2 + 3x + 2}$ cannot lie.

12. Prove that $\dfrac{(2x - 1)(4x - 5)}{(x - 2)(5x - 4)}$ can take any value.

13. For what values of x is $\dfrac{x^2 - 7x - 18}{x^2 + 4x - 21}$ positive? For what values of x is it greater than unity?

14. Show that $\dfrac{3x^2 - 3}{6x - 10}$ has a minimum value 3 and a maximum value $\frac{1}{3}$. Explain the apparent paradox by a rough graph.

15. Prove that, if x is real,

$$\frac{2}{3} \leqslant \frac{x^2+1}{x^2+x+1} \leqslant 2.$$

16. Prove that the minimum value of $\frac{2-x^2}{(1-x)^2}$ is -2, and sketch the graph of this function.

17. If $x^2+y^2-5x-10y=11$ and if x and y have real values, prove that x lies between 9 and -4, and y lies between $11\frac{1}{2}$ and $-1\frac{1}{2}$.

18. If $x^2+4xy+2y^2-8x-12y+15=0$ and if x and y have real values, prove that x cannot lie between 1 and 3.

19. Prove that $\frac{x-a}{x^2-2x+a}$ is capable of all values if $0<a<1$.

20. Find the condition that $\frac{1-x^2}{ax^2+bx+c}$ may be able to assume all values as x varies from $-\infty$ to ∞.

21. If $x=\frac{2t}{1+t^2}$ and $y=\frac{1-t^2}{1+t^2}$, prove that

$$\tfrac{1}{2} \leqslant \frac{7-6x-3y}{9-8x-3y} \leqslant 1.$$

22. Prove that $\frac{x+a}{x^2+bx+c^2}$ will always lie between two fixed limits if $a^2+c^2>ab$ and $b^2<4c^2$; prove also that there will be two limits between which it cannot lie if $a^2+c^2>ab$ and $b^2>4c^2$, and that it is capable of all values if $a^2+c^2<ab$.

23. If a line parallel to the x-axis cuts the graph of

$$y=\frac{x-1}{(x-2)(x-3)}$$

at $x=a$ and $x=b$, prove that $(a-1)(b-1)=2$.

CHAPTER VIII

THEORY OF EQUATIONS

Equations with Given Roots

It is easy to construct an equation which has roots of given values; the degree of the equation equals the number of roots, repeated roots being so counted.

Example 1. Construct the equation whose roots are

(i) $-1, \frac{3}{2}, 5$;

(ii) $-1, \frac{3}{2}$, and 3 roots equal to 5.

(i) The equation may be written

$$(x+1)(2x-3)(x-5)=0.$$

When multiplied out, this becomes an equation of degree 3 in x.

(ii) The equation may be written

$$(x+1)(2x-3)(x-5)^3=0.$$

When multiplied out, this becomes an equation of degree 5 in x.

Example 2. Construct the equation whose roots are

(i) a_1, a_2, a_3; (ii) a_1, a_2, a_3, a_4.

(i) The equation may be written

$$(x-a_1)(x-a_2)(x-a_3)=0.$$

When multiplied out, this becomes

$$x^3-x^2(a_1+a_2+a_3)+x(a_2a_3+a_3a_1+a_1a_2)-a_1a_2a_3=0;$$

and this is expressed more conveniently in the form,

$$x^3-x^2\Sigma(a_1)+x\Sigma(a_1a_2)-a_1a_2a_3=0.$$

(ii) Similarly, the required equation may be written

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4)=0,$$

and, when multiplied out, this becomes

$$x^4-x^3\Sigma(a_1)+x^2\Sigma(a_1a_2)-x\Sigma(a_1a_2a_3)+a_1a_2a_3a_4=0.$$

This example shows the relations which connect the roots of cubic and quartic equations with the coefficients, and suggests the extension to equations of any degree.

Relations between Roots and Coefficients

If $x=a$ is a root of the equation

$$f(x)\equiv p_0x^n+p_1x^{n-1}+\dots+p_{n-1}x+p_n=0,$$

then $f(a)=0$, and therefore, by the remainder theorem, $x-a$ is a factor of $f(x)$.

Hence, if $a_1, a_2, \dots, a_n$ are n unequal roots of this equation, it follows that $x-a_1, x-a_2, \dots, x-a_n$, are each factors of $f(x)$.

$\therefore$ $f(x)\equiv \mathsf{A}(x-a_1)(x-a_2)\dots(x-a_n)$ where A is independent of x since $f(x)$ is of degree n in x; by equating coefficients of x^n, it follows that $\mathsf{A}=p_0$.

$$\therefore\ (x-a_1)(x-a_2)\dots(x-a_n)\equiv x^n+\frac{p_1}{p_0}x^{n-1}+\dots+\frac{p_{n-1}}{p_0}x+\frac{p_n}{p_0}.$$

But the expansion of the left side of this identity is

$$x^n-x^{n-1}\Sigma(a_1)+x^{n-2}\Sigma(a_1a_2)-\dots+(-1)^na_1a_2\dots a_n.$$

$\therefore$ equating coefficients, we have

$$\Sigma(a_1)=-\frac{p_1}{p_0};$$

$$\Sigma(a_1a_2)=\ \frac{p_2}{p_0};$$

$$\Sigma(a_1a_2a_3)=-\frac{p_3}{p_0};$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

and finally, $$a_1a_2a_3\dots a_n=(-1)^n\frac{p_n}{p_0}.$$

These relations express the sums of the products of the roots, taken 1, 2, 3, ... , n, at a time, in terms of the coefficients.

Thus, in particular, if α, β, γ are the roots of the cubic,

$$ax^3+bx^2+cx+d=0,$$

we have $$\alpha+\beta+\gamma=-\frac{b}{a};$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\ \frac{c}{a};$$

$$\alpha\beta\gamma=-\frac{d}{a}.$$

The general relations, established above, remain true if any of the roots $a_1, a_2, \dots$ are equal to one another. Thus if the cubic, $ax^3+bx^2+cx+d=0$, has a root α and two equal roots β, we have

$$\alpha+2\beta=-\frac{b}{a};\ \ \beta^2+2\alpha\beta=\frac{c}{a};\ \ \alpha\beta^2=-\frac{d}{a}.$$

In ordinary (real) algebra, these relations apply only to equations of degree n, if they possess n roots, and the examples of Ch. VII have shown that this is often not the case. In the algebra of complex numbers, it can be proved that an equation of degree n always possesses n roots, and the relations between the roots and the coefficients always hold good. In the examples of this chapter, we shall therefore suppose, where necessary, that we are using the algebra of complex numbers, although such numbers have not been discussed, or even defined. The only defence for this procedure is convenience for examination requirements.

By means of these relations,

Any symmetric function of the roots of an equation can be expressed in terms of the coefficients.

Example 3. If $\alpha, \beta, \gamma, \delta$ are the roots of

$$x^4 - 15x^2 - 10x + 24 = 0,$$

find the values of (i) $\Sigma(\alpha^2)$; (ii) $\Sigma(\alpha^2\beta)$.

We have $\Sigma\alpha = 0$, $\Sigma(\alpha\beta) = -15$, $\Sigma(\alpha\beta\gamma) = 10$.

$$\text{(i) } \Sigma(\alpha^2) = (\Sigma\alpha)^2 - 2\Sigma(\alpha\beta) = 0 + 30 = 30.$$

(ii) Consider the product $\Sigma(\alpha) . \Sigma(\alpha\beta)$.

Terms such as $\gamma^2\alpha$ occur once, and only once, from the product $\gamma . \gamma\alpha$.

Terms such as $\alpha\gamma\delta$ occur as the products,

$$\alpha . \gamma\delta, \ \gamma . \alpha\delta, \ \delta . \alpha\gamma, \text{ that is, 3 times.}$$

$$\therefore \ \Sigma(\alpha) . \Sigma(\alpha\beta) = \Sigma(\alpha^2\beta) + 3\Sigma(\alpha\beta\gamma);$$

$$\therefore \ \Sigma(\alpha^2\beta) = 0 - 3(10) = -30.$$

Note. The reader can check these results by direct calculation, since the roots are $1, -2, -3, 4$.

It is also worth checking identities like

$$\Sigma(\alpha) . \Sigma(\alpha\beta) = \Sigma(\alpha^2\beta) + 3\Sigma(\alpha\beta\gamma)$$

by taking $\alpha = \beta = \gamma = 1$.

Example 4. If α, β, γ are the roots of $px^3 + qx + r = 0$, express, in terms of the coefficients,

$$\text{(i) } \Sigma(\alpha^2); \ \text{(ii) } \Sigma(\alpha^3); \ \text{(iii) } \Sigma(\alpha^5).$$

We have $\Sigma\alpha = 0$, $\Sigma\alpha\beta = \dfrac{q}{p}$, $\alpha\beta\gamma = -\dfrac{r}{p}$.

(i) $\Sigma(\alpha^2)=(\Sigma\alpha)^2-2\Sigma\alpha\beta=0-2\left(\dfrac{q}{p}\right)=-\dfrac{2q}{p}.$

(ii) $p\alpha^3+q\alpha+r=0$, and similar relations hold for β, γ; therefore by addition

$$p\Sigma(\alpha^3)+q\Sigma\alpha+3r=0;$$

$$\therefore\ \Sigma(\alpha^3)=-\frac{3r}{p}.$$

Alternatively,

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma\equiv(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta);$$

$$\therefore\ \alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma=-\frac{3r}{p}\text{ since }\Sigma\alpha=0.$$

(iii) $p\alpha^5+q\alpha^3+r\alpha^2=0$; $\therefore\ \alpha^5=-\dfrac{q}{p}\alpha^3-\dfrac{r}{p}\alpha^2$, and similar relations hold for β, γ; therefore by addition

$$\Sigma(\alpha^5)=-\frac{q}{p}\Sigma(\alpha^3)-\frac{r}{p}\Sigma(\alpha^2)$$

$$=-\frac{q}{p}\left(-\frac{3r}{p}\right)-\frac{r}{p}\left(-\frac{2q}{p}\right)=\frac{5qr}{p^2}.$$

EXERCISE VIII. a

1. Construct the equations whose roots are

(i) 1, 2, 3; (ii) 0, 1, 2, 3; (iii) $\pm1, \pm2, 3$; (iv) $-\frac{1}{2}, -\frac{1}{4}, \frac{3}{4}$.

2. Construct the equation which has 4 roots equal to 1 and 3 roots equal to -1.

3. What general remark can you make about the coefficients of an equation if the roots are

(i) $a, b, c, -a-b-c$; (ii) $\dfrac{a}{b}, \dfrac{b}{c}, \dfrac{c}{a}$;

(iii) $\pm a, \pm b$; (iv) $0, \pm a, \pm b$?

4. If α, β are the roots of $x^2-4x+1=0$, prove that

(i) $\alpha^3+\beta^3=52$; (ii) $\alpha^6+\beta^6=2702$.

5. If α, β, γ are the roots of $x^3-4x^2+2x-1=0$, find the values of (i) $\Sigma\alpha^2$; (ii) $\Sigma\dfrac{1}{\alpha}$; (iii) $\Sigma\alpha^3$.

6. If α, β, γ are the roots of $x^3-x^2+4x-1=0$, find the value of (i) $\Sigma\dfrac{1}{\alpha\beta}$; (ii) $(\alpha+1)(\beta+1)(\gamma+1)$; (ii) $\Sigma(\alpha^2\beta)$.

7. If α, β, γ are the roots of $x^3-3x^2+4=0$, find the value of $\Sigma(\alpha^2)$ and prove that $\Sigma(\alpha^4)=33$.

8. Find the sum of the fourth powers of the roots of

$$3x^4 - 6x^2 + 4 = 0.$$

9. If α, β, γ are the roots of $ax^3 + cx + d = 0$, find, in terms of the coefficients, (i) $\Sigma \frac{1}{\alpha}$; (ii) $\Sigma \frac{1}{\alpha^2}$.

10. If α, β, γ are the roots of $ax^3 + bx^2 + d = 0$, find, in terms of the coefficients, (i) $\Sigma(\alpha^2)$; (ii) $\Sigma \frac{1}{\alpha\beta}$.

11. If α, β, γ are the roots of $x^3 + px + q = 0$, find, in terms of the coefficients, the values of (i) $\Sigma(\alpha^4)$; (ii) $\Sigma(\alpha^2\beta)$; (iii) $\Sigma(\alpha^2\beta^2)$.

12. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find, in terms of the coefficients, (i) $\Sigma(\alpha^2)$; (ii) $\Sigma \frac{1}{\alpha}$; (iii) $\Sigma(\alpha^2\beta)$.

13. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find, in terms of the coefficients, (i) $\Sigma(\alpha - \beta)^2$; (ii) $\Sigma\alpha(\beta - \gamma)^2$.

14. If α, β, γ are the roots of $x^3 + px + q = 0$, find, in terms of the coefficients,

(i) $(\alpha + \beta - 2\gamma)(\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)$; (ii) $\Sigma\alpha^3(\beta + \gamma)$.

15. If α, β, γ, δ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find, in terms of the coefficients,

(i) $\Sigma(\alpha^2\beta^2\gamma^2)$; (ii) $\Sigma\alpha\beta\gamma(\alpha + \beta + \gamma)$.

16. Find the sum of the cubes of the roots of $x^5 = x^2 + x + 1$.

17. Prove that the sum of the eleventh powers of the roots of $x^5 + 5x + 1 = 0$ is zero.

18. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots = 0$, prove that $\Sigma(\alpha_1^3) = 3p_1p_2 - p_1^3 - 3p_3$.

Unsymmetrical Relations between the Roots

The symmetrical relations between the roots of an equation and the coefficients do not by themselves provide a means of solving the equation; e.g. the elimination of β, γ from

$$\alpha + \beta + \gamma = -\frac{b}{a}, \quad \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}$$

merely leads back to the original equation, $ax^3 + bx^2 + cx + d = 0$, with α instead of x. In fact the roots of the general equation of the nth degree cannot be expressed by elementary functions of the coefficients if $n > 4$. But if some additional unsymmetrical relation is known to exist between two or more of the roots, it may be possible to solve the equation or to find some of its roots.

Example 5. Solve $12x^3 - 28x^2 + 3x + 18 = 0$, given that it has two equal roots.

Suppose the roots are α, α, β.

Then $2\alpha + \beta = \frac{7}{3}$, $\alpha^2 + 2\alpha\beta = \frac{1}{4}$, $\alpha^2\beta = -\frac{3}{2}$.

$\therefore\ \alpha^2 + 2\alpha(\frac{7}{3} - 2\alpha) = \frac{1}{4}$; and this reduces to

$$36\alpha^2 - 56\alpha + 3 = 0 \text{ or } (2\alpha - 3)(18\alpha - 1) = 0.$$

Hence $\alpha = \frac{3}{2}$, $\beta = -\frac{2}{3}$ or $\alpha = \frac{1}{18}$, $\beta = \frac{20}{9}$.

Also $\alpha^2\beta = -\frac{3}{2}$ is satisfied by the first pair of values of α and β, but not by the second.

$$\therefore \text{ the roots are } \tfrac{3}{2}, \tfrac{3}{2}, -\tfrac{2}{3}.$$

Note. The reader should compare this method with that given for the same equation on p. 143.

Example 6. Solve $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in A.P.

Suppose the roots are $a - b, a, a + b$.

Then $(a - b) + a + (a + b) \equiv 3a = 6$,

$(a - b)a + a(a + b) + (a + b)(a - b) \equiv 3a^2 - b^2 = \frac{23}{4}$, $a(a^2 - b^2) = -\frac{9}{2}$.

$\therefore\ a = 2$ and $b^2 = \frac{25}{4}$; $\therefore\ b = \pm\frac{5}{2}$; $\therefore$ the roots are $-\frac{1}{2}, 2, 4\frac{1}{2}$.

Example 7. Find the condition that the roots of

$$x^3 + px^2 + qx + r = 0$$

are in G.P.

Suppose the roots are α, β, γ where $\beta^2 = \alpha\gamma$. Then

$$\alpha + \beta + \gamma = -p, \quad \beta\gamma + \gamma\alpha + \alpha\beta = q, \quad \alpha\beta\gamma = -r.$$

$$\therefore\ q = \beta\gamma + \beta^2 + \alpha\beta = \beta(\gamma + \beta + \alpha) = -p\beta.$$

Also $r = -\alpha\beta\gamma = -\beta^3 = -\left(-\frac{q}{p}\right)^3$; $\therefore\ rp^3 = q^3$.

EXERCISE VIII. b

1. Solve $x^3 - 6x^2 + 3x + 10 = 0$, given the roots are in A.P.

2. Solve $2x^3 - 21x^2 + 42x - 16 = 0$, given the roots are in G.P.

3. Solve $x^3 - 7x^2 + 36 = 0$, given that one root is double another

4. Solve $x^3 - 5x^2 - 2x + 24 = 0$, given that the product of two of the roots is 12.

5. Solve $3x^3 - 7x^2 - 36x - 20 = 0$, given that the sum of two of the roots is 3.

6. Solve $x^4+x^3-16x^2-4x+48=0$, given that the product of two of the roots is 6.

7. Solve $x^3-4x^2-3x+18=0$, given that two of the roots are equal.

8. Solve $x^4-11x^3+28x^2+36x=144$, given that the product of two of the roots and the product of the other two differ only in sign.

9. Solve the equation $x^3-5x^2-2x+24=0$, given that one root exceeds another by 1.

10. Write down the equation whose roots are a, b, c, where $a+b+c=2$, $bc+ca+ab=-5$, $abc=-6$. Hence solve these simultaneous equations.

11. Solve the simultaneous equations,

$$a+b+c=-1,\ a^2+b^2+c^2=9,\ abc=4.$$

12. Solve the simultaneous equations,

$$x+y+z=1,\ x^2+y^2+z^2=29,\ x^3+y^3+z^3=-29.$$

13. Find the condition that two of the roots of

$$x^3+px^2+qx+r=0$$

differ only in sign.

14. If the roots of $x^3+px^2+qx+r=0$ are in A.P., prove that $2p^3+27r=9pq$.

15. If $x^3+qx+r=0$ has two equal roots, prove that

(i) $4q^3+27r^2=0$, (ii) the repeated root equals $-\dfrac{3r}{2q}$.

16. If one of the roots of $ax^2+bx+c=0$ is the cube of the reciprocal of the other, prove that $(a^2+c^2)^2=ab^2c$.

17. If α, β are the roots of $x^2+px+q=0$, find in terms of α, β, the roots of $qx^2+(2q-p^2)x+q=0$.

18. If one of the roots of $x^4+px^3+qx+r=0$ equals the sum of the other three, prove that $p^4+8pq=16r$.

19. If the product of two of the roots of

$$x^4+px^3+qx^2+rx+s=0$$

equals the product of the other two, prove that $r^2=p^2s$.

20. If the sum of two of the roots of $x^4+px^3+qx^2+rx+s=0$ equals the sum of the other two, prove that $p^3+8r=4pq$.

21. If $x^4-px^2+qx-r=0$ has three equal roots, prove that

(i) $p^2=12r$ and $9q^2=32pr$, (ii) the repeated root equals $\dfrac{3q}{4p}$.

Transformation of Equations

Properties of the roots of an equation are often best investigated by constructing another equation whose roots are related to those of the given equation by some law.

For simplicity, we shall use cubic equations to illustrate the chief transformations, but the methods are general.

To form the equation whose roots are (i) the reciprocals of the roots (ii) k times the roots of $ax^3+bx^2+cx+d=0$.

(i) If α, β, γ are the roots of $ax^3+bx^2+cx+d=0$, and if $y=\frac{1}{x}$, then y has one of the values $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, when

$$ax^3+bx^2+cx+d=0.$$

If $y=\frac{1}{x}$, then $x=\frac{1}{y}$, $\therefore \frac{a}{y^3}+\frac{b}{y^2}+\frac{c}{y}+d=0$.

$$\therefore \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma} \text{ are the roots of } dy^3+cy^2+by+a=0.$$

(ii) Similarly, if $y=kx$, so that $x=\frac{y}{k}$, then y has one of the values $k\alpha$, $k\beta$, $k\gamma$, if $a\left(\frac{y}{k}\right)^3+b\left(\frac{y}{k}\right)^2+c\left(\frac{y}{k}\right)+d=0$, that is, if $ay^3+bky^2+ck^2y+dk^3=0$.

This process is called "*multiplying the roots* of an equation by k."

Thus, the equation whose roots are 10 times the roots of $x^3+3x^2+6x-3=0$ is $y^3+30y^2+600y-3000=0$.

To form the equation whose roots are respectively less by k than the roots of $ax^3+bx^2+cx+d=0$.

If α, β, γ are the roots of $ax^3+bx^2+cx+d=0$, and if $y=x-k$, then y has one of the values $\alpha-k$, $\beta-k$, $\gamma-k$ if

$$ax^3+bx^2+cx+d=0.$$

But $x=y+k$,

$$\therefore a(y+k)^3+b(y+k)^2+c(y+k)+d=0$$

is the required equation.

This process is called "*diminishing the roots* of an equation by k."

In practice, this transformation is made by expressing

$$ax^3+bx^2+cx+d$$

in the form $a(x-k)^3+p(x-k)^2+q(x-k)+r$; and this is done by successive division by $(x-k)$, as illustrated by the following example :

Example 8. Diminish by 2 the roots of $2x^3-7x^2+3x-5=0$.

$$2x^3-7x^2+3x-5=(x-2)(2x^2-3x-3)-11,$$

$$2x^2-3x-3=(x-2)(2x+1)-1,$$

$$2x+1=(x-2)2+5,$$

$$\therefore\ 2x^3-7x^2+3x-5=2(x-2)^3+5(x-2)^2-(x-2)-11.$$

The successive division may be worked as follows:

$$\begin{array}{cccc}
2 & -7 & +3 & -5 \\
 & \underline{+4} & \underline{-6} & \underline{-6} \\
 & -3 & -3 & \mathbf{-11} \\
 & \underline{+4} & \underline{+2} & \\
 & +1 & \mathbf{-1} & \\
 & \underline{+4} & & \\
 & \mathbf{+5} & &
\end{array}$$

If the reader compares this with the corresponding long division method, set out in full but using detached coefficients (*New Algebra*, Part III, pp. 107, 108), he will find that it contains the whole of the working, arranged concisely, the successive quotients being

$$2\ -3\ -3;\quad 2\ +1;\quad 2.$$

If there is occasion to use this process frequently, there is no difficulty in abbreviating it, as shown here. It is employed in Horner's method for solving numerical equations, see p. 165.

$$\begin{array}{cccc}
2 & -7 & +3 & \underline{-5} \\
 & -3 & \underline{-3} & \mathbf{-11} \\
 & \underline{+1} & \mathbf{-1} & \\
 & \mathbf{+5} & &
\end{array}$$

Example 9. If α, β, γ are the roots of $x^3+qx+r=0$, form the equation whose roots are $\beta+\gamma-2\alpha$, $\gamma+\alpha-2\beta$, $\alpha+\beta-2\gamma$.

Since $\alpha+\beta+\gamma=0$, we require the equation whose roots are

$$-3\alpha,\ -3\beta,\ -3\gamma.$$

Put $y=-3x$, that is, $x=-\frac{1}{3}y$;

$$\therefore\ -\tfrac{1}{27}y^3-\tfrac{1}{3}qy+r=0;$$

$\therefore$ the required equation is $y^3+9qy-27r=0$.

Example 10. If α, β, γ are the roots of $x^3+2x^2-5x-6=0$, form the equation whose roots are $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$.

We have $\Sigma\alpha\beta = -5$, $\alpha\beta\gamma=6$,

$$\therefore\ \alpha(\beta+\gamma) = -5-\beta\gamma = -5-\frac{6}{\alpha}.$$

$\therefore$ if $y=-5-\dfrac{6}{x}$, y takes the required values when x takes the values α, β, γ; this gives $x=-\dfrac{6}{y+5}$;

$$\therefore\ -\left(\frac{6}{y+5}\right)^3+2\left(\frac{6}{y+5}\right)^2+5\left(\frac{6}{y+5}\right)-6=0,$$

which reduces to $y^3+10y^2+13y-24=0$.

Note. The reader can check this result by direct calculation, because the roots of the original equation are 2, -1, -3.

EXERCISE VIII. c

1. If α, β, γ are the roots of $x^3+2x^2=2$, form the equation whose roots are

   (i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$; (ii) 10α, 10β, 10γ; (iii) $\alpha-1$, $\beta-1$, $\gamma-1$.

2. If α, β, γ are the roots of $x^3+3x=3$, form the equation whose roots are

   (i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$; (ii) $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$; (iii) $\alpha+1$, $\beta+1$, $\gamma+1$.

3. With the data of No. 2, form the equation whose roots are (i) $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$; (ii) $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

4. If α, β, γ, δ are the roots of $x^4-x^2-3x-5=0$, form the equation whose roots are

   (i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}, \dfrac{1}{\delta}$; (ii) -2α, -2β, -2γ, -2δ.

5. Increase the roots of $x^4+4x^3-7x^2-22x+24=0$ by 1; hence solve this equation.

6. Given that the roots of $x^3-x^2-14x+24=0$ are rational, explain why they must be integral and factors of 24. Hence solve the equation.

7. Given that the roots of $6x^3+19x^2+x-6=0$ are rational, multiply the roots by some constant k so as to obtain an equation whose roots are integral. Hence solve the equation.

8. Use the transformation $x^2+3x+3=y$ to solve

$$(x^2+3x+3)^2=2x^2+6x+5.$$

9. Diminish the roots of $x^3 - 9x^2 + 28x - 27 = 0$ by 3; hence prove that this equation has no root greater than 3 and only one root less than 3.

10. One root of $x^3 + x^2 - 24x + 16 = 0$ is an integer. Find the other two roots correct to 2 places of decimals.

11. If α, β are the roots of $(x-a)(b-x) = h^2$, prove that $a + 2\beta$, $\beta + 2a$ are the roots of $(x - a - 2b)(2a + b - x) = h^2$.

12. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - px^3 + q = 0$, form the equation whose roots are $\alpha + \beta + \gamma$, $\alpha + \beta + \delta$, $\alpha + \gamma + \delta$, $\beta + \gamma + \delta$.

13. Prove that the equation $x^3 - 3px^2 + 3qx - r = 0$ may be reduced to the form $y^3 + 3by + c = 0$ by diminishing the roots by a suitable (positive or negative) constant. Apply the transformation to $x^3 + 6x^2 + 9x + 4 = 0$.

14. Diminish the roots of the equation

$$(m+2)x^2 - 4mx + m - 1 = 0$$

by 3. Hence find for what values of m both roots of the given equation are less than 3.

15. If α, β are the roots of $x^2 + qx + r = 0$, form the equation whose roots are $(\alpha - 1)^2$, $(\beta - 1)^2$.

16. If α, β, γ are the roots of $x^3 + qx + r = 0$, form the equation whose roots are $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}, \frac{1-\gamma}{\gamma}$.

17. If α, β, γ are the roots of $x^3 + qx + r = 0$, form the equation whose roots are (i) $\alpha^2, \beta^2, \gamma^2$; (ii) $\frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}, \frac{\alpha\beta}{\gamma}$.

18. Solve $(x+b+c)(x+c+a)(x+a+b) + abc = 0$ by means of the substitution $x + a + b + c = y$.

19. If $y = x + \frac{1}{x}$, express $x^2 + \frac{1}{x^2}$ in terms of y. Hence solve $x^4 - 2x^3 - 6x^2 - 2x + 1 = 0$. [Divide by x^2.]

20. Use the transformation $x + \frac{1}{x} = y$ to solve

$$x^4 - 5x^3 - 12x^2 - 5x + 1 = 0.$$

21. Solve $x^3 - 11x^2 + 37x - 35 = 0$, given that one root is $3 + \sqrt{2}$.

22. If α is a root of $x^4 + x^3 - 4x^2 - 4x + 1 = 0$, prove that $2 - \alpha^2$ is another root.

23. Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$, given that one root is $2 - \sqrt{3}$.

24. Diminish the roots of $2x^3 - 15x^2 + 31x - 12 = 0$ by 1. Hence solve the equation, given that two of the roots differ by 1.

25. Increase by 1 each of the roots of the simultaneous equations, $xy+x+y=-5$, $yz+y+z=7$, $zx+z+x=-3$, and hence solve them.

26. If both roots of $ax^2-bx+c=0$ are greater than 1, prove that a, $a-b+c$, $b-2a$ have the same signs.

27. Prove that $x=\dfrac{1}{1-a}$ is a root of the equation

$$x+\frac{1}{1-x}+\frac{x-1}{x}=a+\frac{1}{1-a}+\frac{a-1}{a},$$

and find the other roots.

28. If α, β, γ are the roots of $x^3+px^2+qx+r=0$, form the equation whose roots are $\alpha^2-\beta\gamma$, $\beta^2-\gamma\alpha$, $\gamma^2-\alpha\beta$. Explain the special results when (i) $p=0$, (ii) $q=0$.

Numerical Equations

There are several ways of obtaining approximate values of the irrational roots of equations having numerical coefficients; we shall illustrate two of these methods by examples.

Newton's Method of Successive Approximation

Example 11. Find an approximate solution of

$$x^3+3x-7=0.$$

If $f(x)\equiv x^3+3x-7$, $f(1)=-3$ and $f(2)=7$;

$$\therefore\ f(x)=0 \text{ has one root between 1 and 2.}$$

Put $x=1+h$, $\therefore\ (1+h)^3+3(1+h)-7=0$.

As a first approximation, neglect h^2, h^3.

Then $$1+3h+3+3h-7=0;$$

$$\therefore\ 6h=3;\quad \therefore\ h=0{\cdot}5.$$

Therefore a first approximation is $x=1{\cdot}5$.

Put $x=1{\cdot}5+k$, $\therefore\ (1{\cdot}5+k)^3+3(1{\cdot}5+k)-7=0$.

As before, neglect k^2, k^3.

Then $(1{\cdot}5)^3+3k(1{\cdot}5)^2+4{\cdot}5+3k-7=0$.

Hence $9{\cdot}75k=-0{\cdot}875$, $\therefore\ k\simeq-0{\cdot}09$.

Therefore to a second approximation, $x=1{\cdot}41$. Closer approximations can be found by repeating the process.

Note. The method of Example 3, p. 141, shows that there is only one root.

Horner's Method

Example 12. Find an approximate solution of

$$x^3+3x-7=0.$$

As in Example 11, we see that there is a root between 1 and 2. If, then, we diminish the roots of the equation by 1, we obtain a new equation with a root between 0 and 1.

The working may be set out as on p. 161.

$$\begin{array}{cccc}
1 & 0 & 3 & -7 \\
 & 1 & \underline{4} & \overline{-3} \\
 & \underline{2} & 6 & \\
 & 3 & &
\end{array}$$

Therefore the transformed equation is

$$x^3+3x^2+6x-3=0.$$

Multiply the roots of this equation by 10; it follows from p. 160 that the transformed equation

$$x^3+30x^2+600x-3000=0$$

has a root between 0 and 10. By trial, we find that this root lies between 4 and 5.

Diminish the roots of the equation by 4.

$$\begin{array}{cccc}
1 & 30 & 600 & -3000 \\
 & 34 & \underline{736} & \underline{2944} \\
 & \underline{38} & 888 & -56 \\
 & 42 & &
\end{array}$$

$\therefore$ the transformed equation

$$x^3+42x^2+888x-56=0$$

has a root between 0 and 1.

Multiplying the roots by 10, it follows that

$$x^3+420x^2+88800x-56000=0$$

has a root between 0 and 10. By trial, we find that this root lies between 0 and 1.

The process can then be continued as before, but it is evident that a good approximation is given by

$$88800x-56000=0, \text{ that is, } x \simeq 0{\cdot}6.$$

$\therefore$ a root of the original equation is $x \simeq 1{\cdot}406$.

The whole process may be set out thus :

1	0	3	-7	(1·406
	1	4	-3	
	2	6		
1	30	600	-3000	
	34	736	-56	
	38	888		
1	420	88800	-56000	

or approximately,

0	0	88	-56

If four places of decimals are required, replace the last line by

0	4	888	-560

and continue as at first.

If *Horner's method* is used to find the approximate value of a **negative root**, the equation should first be transformed by putting $x=-y$; the method is then applied to find the corresponding positive root of the transformed equation.

EXERCISE VIII. d

Find approximate solutions of the following equations :

1. $x^3-2=0$.
2. $x^3-2x-5=0$.
3. $x^3-5x+3=0$, (3 roots).
4. $x^3=7x+1$, (3 roots).
5. $2x^3-3x-6=0$.
6. $x^3-9x+14=0$.
7. $x^3=8x+1$, (3 roots).
8. $x^3+x=1000$.
9. $x^4-12x+7=0$.
10. $x^3-3x^2+5x=43$.
11. $x^3-8x^2-6x+9=0$, (3 roots).
12. $x^5-7=0$.
13. $x^3+2x^2-5x-7=0$, positive root only.
14. $x^3-7x+7=0$, two roots between 1 and 2.
15. $x^5=4x+2000$.

Elimination

Example 13. Find the condition that the equations,

$$a_1x^2+b_1x+c_1=0, \quad a_2x^2+b_2x+c_2=0,$$

may be simultaneously true for some value of x.

If there is a value of x which satisfies both equations, we have by solving, see *New Algebra*, Part III, p. 127,

$$\frac{x^2}{b_1c_2-b_2c_1}=\frac{x}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

$$\therefore\ (c_1a_2-c_2a_1)^2=(b_1c_2-b_2c_1)(a_1b_2-a_2b_1).$$

This process is called **eliminating** x from the two given equations.

Example 14. Eliminate p, q, r from the equations

$$y+px=2ap+ap^3,\ y+qx=2aq+aq^3,\ y+rx=2ar+ar^3,$$

and $pq=b$.

p, q, r are the roots of the cubic equation,

$$at^3+t(2a-x)-y=0.$$

$$\therefore\ p+q+r=0,\ pq+qr+rp=\frac{2a-x}{a},\ pqr=\frac{y}{a};$$

$$\therefore\ r=\frac{y}{apq}=\frac{y}{ab}\ \text{and}\ p+q=-r=-\frac{y}{ab};$$

$$\therefore\ \frac{2a-x}{a}=pq+r(p+q)=b-\frac{y^2}{a^2b^2}.$$

$$\therefore\ y^2=ab^2x+a^2b^2(b-2).$$

EXERCISE VIII. e

1. Find the relation between a, b, c if values of x and y exist such that $x+ay=1$, $x+by=2$, $x+cy=3$.

2. If $(x+y)^2=a^2+b^2$, $a^2=x^2+h^2$, $b^2=y^2+h^2$, prove that $xy=h^2$ and interpret the result geometrically.

Eliminate t from the equations in Nos. 3-8.

3. $x=at^2$, $y=2at$.

4. $x=t+\dfrac{1}{t}$, $y=t^2+\dfrac{1}{t^2}$.

5. $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{2bt}{1+t^2}$.

6. $x=\dfrac{3at}{1+t^3}$, $y=\dfrac{3at^2}{1+t^3}$.

7. $x=a+bt+ct^2$, $y=b+ct$.

8. $\dfrac{x}{t^3}=\dfrac{y}{t^2}=z$.

9. Eliminate x, y, z, given

$$\Sigma x=a,\ \Sigma(xy)=b^2,\ \Sigma(x^3)=c^3,\ xyz=d^3.$$

10. Eliminate x, y, z, given

$$xy=a^2,\ yz=b^2,\ zx=c^2,\ x^2+y^2+z^2=d^2.$$

11. Eliminate x from the equations,

$$ax^2+bx+c=cx^2+bx+a=d, \ c \neq a.$$

12. If $a^2+x^2=b^2+y^2=ay-bx=1$, prove that $a^2+b^2=1$.

13. If $\frac{a}{x}+\frac{b}{y}=1$, $\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$, $xy=a^2+b^2$, prove that $a+b=0$.

14. If $x^3+2px^2+2qx+r=0$ and $x^2+px+q=0$ have a common root, prove that
$$r^2-3pqr+p^3r+q^3=0.$$

15. If $x^3+ax^2+b=0$ and $x^2+lx+m=0$ have a common root, prove that

$$m^3+a(a-l)m^2+b(3l-2a)m+b(b+al^2-l^3)=0.$$

16. If $x=\alpha$ is a common root of the equations,

$$2kpx^3-(1+p^2)cx+b(1-p^2)=0,$$
$$2kqx^3-(1+q^2)cx+b(1-q^2)=0,$$

prove that $\frac{p+q}{2k\alpha^3}=\frac{pq}{c\alpha-b}=\frac{1}{c\alpha+b}$.

Hence show that $kb^2(1+pq)^3=c^3(p+q)(1-pq)^2$.

17. If $m^2x-my+a=0=n^2x-ny+a$, and if $m-n=c(1+mn)$, prove that
$$y^2-4ax=c^2(x+a)^2.$$

18. If $ax-by=x^2-y^2$, $ay+bx=4xy$, $x^2+y^2=1$, prove that

$$(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2.$$

19. Eliminate m and n from the equations,

$$(y-mx)^2=a^2m^2+b^2, \ (y-nx)^2=a^2n^2+b^2, \ mn=-1.$$

20. Eliminate x and y from the equations, $y+mx=am^3+2am$, $y+nx=an^3+2an$, $y+px=ap^3+2ap$.

21. Eliminate λ, μ, ν from the relations,

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \quad \frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}+\frac{z^2}{c^2+\mu}=1,$$
$$\frac{x^2}{a^2+\nu}+\frac{y^2}{b^2+\nu}+\frac{z^2}{c^2+\nu}=1, \quad \lambda\mu\nu=a^2b^2c^2.$$

CHAPTER IX

DETERMINANTS

The solution of the equations,

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0,$$

may be written, see *New Algebra*, Part III, p. 127

$$\frac{x}{b_1c_2-c_1b_2}=\frac{y}{c_1a_2-a_1c_2}=\frac{1}{a_1b_2-b_1a_2},$$

provided that $a_1b_2-b_1a_2\neq 0$.

It is often convenient to denote expressions such as $a_1b_2-b_1a_2$ by symbols like $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

This arrangement is called a **determinant of the second order.**

The solution of the given simultaneous equations can then be put in the form,

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

where the denominators are the determinants obtained from

$$\left\|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\|$$

by omitting in turn the first, second and third columns.

In this chapter, we shall develop this notation, but some initial practice with simple forms will help the reader to understand the extensions.

In the determinant,

$$\Delta\equiv\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\equiv +a_1b_2-b_1a_2,$$

the numbers are arranged in two *rows*, a_1, b_1 and a_2, b_2, and in two *columns*, a_1, a_2 and b_1, b_2; and the numbers themselves are called *elements* or *constituents* of Δ. Each term in the expansion

of Δ is formed by 2 factors which belong to different rows and to different columns.

In this example, the suffixes 1, 2, show the row to which the element belongs, and the letters a, b, show the column.

The value of Δ is unaltered if columns and rows are interchanged.

$$\text{Thus}\quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = +a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

In the new form, suffixes 1, 2, indicate columns, and letters a, b, indicate rows. Since the interchange of rows and columns does not affect Δ, *any property established about "rows" must also hold for "columns."*

If two rows of Δ are interchanged, the numerical value of Δ is unaltered, but its sign is changed.

$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = +a_2b_1 - b_2a_1 = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Similarly for columns, $\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$

If two rows (or two columns) of Δ are identical, the value of Δ is zero.

$$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = +a_1b_1 - a_1b_1 = 0.$$

If all the elements of one row (or of one column) of Δ are multiplied by k, the new determinant equals $k\Delta$.

$$\begin{vmatrix} a_1 & b_1 \\ ka_2 & kb_2 \end{vmatrix} = +a_1kb_2 - b_1ka_2 = k \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

$$\text{Thus}\quad \begin{vmatrix} 63 & 27 \\ 35 & 16 \end{vmatrix} = 9 \times \begin{vmatrix} 7 & 3 \\ 35 & 16 \end{vmatrix} = 9 \times 7 \times \begin{vmatrix} 1 & 3 \\ 5 & 16 \end{vmatrix}.$$

$$= 63(16 - 15) = 63.$$

A determinant of the form $\begin{vmatrix} a_1 + x_1 & b_1 + y_1 \\ a_2 & b_2 \end{vmatrix}$ is the sum of the two determinants, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\begin{vmatrix} x_1 & y_1 \\ a_2 & b_2 \end{vmatrix}$

since $\quad (a_1 + x_1)b_2 - (b_1 + y_1)a_2 = (a_1b_2 - b_1a_2) + (x_1b_2 - y_1a_2).$

Consequently, $\begin{vmatrix} a_1+ka_2 & b_1+kb_2 \\ a_2 & b_2 \end{vmatrix}$

$$=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}+k\begin{vmatrix} a_2 & b_2 \\ a_2 & b_2 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This establishes the following important property:

The value of a determinant is not altered by adding to the elements of a row (or column) the same (positive or negative) multiples of the corresponding elements of the other row (or column).

This fact is often of assistance in evaluating determinants.

Example 1. Evaluate $\Delta\equiv\begin{vmatrix} 103 & 259 \\ 33 & 83 \end{vmatrix}$.

$$\Delta=\begin{vmatrix} 103-(3\times 33) & 259-(3\times 83) \\ 33 & 83 \end{vmatrix}=\begin{vmatrix} 4 & 10 \\ 33 & 83 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 10 \\ 33-(8\times 4) & 83-(8\times 10) \end{vmatrix}=\begin{vmatrix} 4 & 10 \\ 1 & 3 \end{vmatrix}=12-10=2.$$

This means that $103\times 83-259\times 33=2$.

Example 2. Evaluate $\Delta\equiv\begin{vmatrix} 560 & 170 \\ 387 & 117 \end{vmatrix}$.

$$\Delta=10\times 9\times\begin{vmatrix} 56 & 17 \\ 43 & 13 \end{vmatrix}=90\times\begin{vmatrix} 56-(3\times 17) & 17 \\ 43-(3\times 13) & 13 \end{vmatrix}$$

$$=90\times\begin{vmatrix} 5 & 17 \\ 4 & 13 \end{vmatrix}=90(65-68)=-270.$$

EXERCISE IX. a

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 1 \\ 7 & 3 \end{vmatrix}$. 2. $\begin{vmatrix} 0 & 2 \\ 5 & 7 \end{vmatrix}$. 3. $\begin{vmatrix} 90 & 80 \\ 70 & 60 \end{vmatrix}$.

4. $\begin{vmatrix} 45 & 37 \\ 45 & 17 \end{vmatrix}$. 5. $\begin{vmatrix} 28 & 29 \\ 30 & 31 \end{vmatrix}$. 6. $\begin{vmatrix} 21 & 36 \\ 28 & 45 \end{vmatrix}$.

7. $\begin{vmatrix} 19 & 39 \\ 23 & 48 \end{vmatrix}$. 8. $\begin{vmatrix} 203 & 305 \\ 99 & 152 \end{vmatrix}$. 9. $\begin{vmatrix} 1931 & 1932 \\ 1933 & 1934 \end{vmatrix}$.

Expand the following determinants:

10. $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$. 11. $\begin{vmatrix} a & b \\ a^3 & b^3 \end{vmatrix}$. 12. $\begin{vmatrix} x+y & x+2y \\ x-y & x-2y \end{vmatrix}$.

13. $\begin{vmatrix} a^2-b^2 & (a-b)^2 \\ a^2+ab & ab-b^2 \end{vmatrix}$.   14. $\begin{vmatrix} px+qz & py+qz \\ rx+sz & ry+sz \end{vmatrix}$.

Solve the following equations, paired as in Nos. 15-20 :

(i) $2x+3y+7=0$; (ii) $3x-2y-5=0$;
(iii) $2x-5y+3=0$; (iv) $3x+4y-7=0$.

15. (i) and (ii). 16. (i) and (iii). 17. (i) and (iv).
18. (ii) and (iii). 19. (ii) and (iv). 20. (iii) and (iv).

Example 3. Find the condition that the equations,

$$a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0, \quad a_3x+b_3y+c_3=0,$$

are satisfied by the same values of x and y.

Solving the last two equations, we have

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}},$$

provided that $a_2b_3-b_2a_3 \neq 0$. If $a_2b_3-b_2a_3=0$, the equations are inconsistent unless $\frac{a_2}{a_3}=\frac{b_2}{b_3}=\frac{c_2}{c_3}$, in which case they are identical.

These values of x and y satisfy the first equation if

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0,$$

and this is written in the form,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This arrangement of symbols is called a **determinant of the third order**, or a determinant with *three rows and three columns*.

By definition,

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= +a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= +a_1(b_2c_3-c_2b_3) - b_1(a_2c_3-c_2a_3) + c_1(a_2b_3-b_2a_3).$$

If we re-arrange this expression, grouping it by elements of the first *column*, we have

$$\Delta = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= +a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Thus we have now extended to determinants of the third order the important fact:

The value of a determinant is unaltered if columns and rows are interchanged.

Since the interchange of rows and columns does not affect the value of Δ, **any property established for "rows" of a third order determinant must also hold for "columns."**

If we re-arrange the expression for Δ, grouping it by elements of the second row, we have

$$\Delta = -a_2(b_1c_3 - c_1b_3) + b_2(a_1c_3 - c_1a_3) - c_2(a_1b_3 - b_1a_3);$$

$$\therefore\ \Delta = -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix},$$

and grouping it by elements of the third row we have

$$\Delta = +a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The interchange of two rows is equivalent to interchanging two suffixes, and it is evident from these expressions for Δ that the effect is to change the sign of the determinant. This is stated as follows:

If two rows, or two columns, of a determinant are interchanged, the numerical value is unaltered but the sign is changed.

$$\text{Thus}\quad \begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ 7 & 5 & 3 \\ 5 & 8 & 2 \end{vmatrix} = -(-1)(7\times 2 - 3\times 5) = -1.$$

It should be noted that each term in the expansion of Δ is formed by 3 factors which belong to different rows and to different

columns ; sets of such factors can be formed in 3×2 ways, giving 6 terms in all. The term $+a_1b_2c_3$ given by the diagonal which slopes down from left to right is called the "*leading term.*" By interchanging 2 suffixes (*or* 2 letters) and changing the sign, the other terms can be obtained. Thus interchanging 2, 3, $+a_1b_2c_3$ leads to $-a_1b_3c_2$, then interchanging 1, 3 we have $+a_3b_1c_2$, and so on.

If two rows, or two columns, of a determinant are identical, the value of the determinant is zero.

If two rows are interchanged, the sign of the determinant is changed. But if the two rows are identical, the interchange makes no difference.

$$\therefore\ \Delta = -\Delta\ ;\ \therefore\ \Delta = 0.$$

$$\text{Thus}\quad \begin{vmatrix} 17 & 9 & 17 \\ 9 & 2 & 9 \\ 15 & 4 & 15 \end{vmatrix} = 0.$$

If all the elements of one row, or of one column, are multiplied by the same constant, the value of the determinant is multiplied by that constant.

Each term in the expansion contains one element, and only one, from each row. Therefore if each element in a row is multiplied by k, say, each term in the expansion is also multiplied by k.

$$\text{Thus}\quad \begin{vmatrix} 56 & 9 & 0 \\ 24 & 15 & 18 \\ 16 & 7 & 5 \end{vmatrix} = 8 \times \begin{vmatrix} 7 & 9 & 0 \\ 3 & 15 & 18 \\ 2 & 7 & 5 \end{vmatrix} = 24 \times \begin{vmatrix} 7 & 9 & 0 \\ 1 & 5 & 6 \\ 2 & 7 & 5 \end{vmatrix}.$$

Minors and Cofactors

The determinant obtained by deleting from Δ the row and column which passes through any element is called the **minor** of that element in Δ.

Thus the minors of a_1, b_1, c_1 in

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

are
$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},\quad \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix},\quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix};$$

and these appear in the expansion

$$\Delta = +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Similarly, the minors of a_2, b_2, c_2 are

$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$; and these enter into

$$\Delta = -a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

Similarly, the minors of a_3, b_3, c_3 enter into

$$\Delta = +a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

It is, however, usually more convenient to write Δ in the form,

$$\Delta = +a_1\mathsf{A}_1 + b_1\mathsf{B}_1 + c_1\mathsf{C}_1,$$

where $\mathsf{A}_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $\mathsf{B}_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $\mathsf{C}_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

A_1, B_1, C_1 are then called the **cofactors** of a_1, b_1, c_1 in Δ.

Similarly, if $\Delta = +a_2\mathsf{A}_2 + b_2\mathsf{B}_2 + c_2\mathsf{C}_2$, A_2, B_2, C_2 are called the *cofactors* of a_2, b_2, c_2 in Δ, and if $\Delta = +a_3\mathsf{A}_3 + b_3\mathsf{B}_3 + c_3\mathsf{C}_3$, A_3, B_3, C_3 are called the *cofactors* of a_3, b_3, c_3 in Δ.

Since a determinant is unaltered when rows and columns are interchanged, we have also

$$\Delta = +a_1\mathsf{A}_1 + a_2\mathsf{A}_2 + a_3\mathsf{A}_3 = +b_1\mathsf{B}_1 + b_2\mathsf{B}_2 + b_3\mathsf{B}_3$$
$$= +c_1\mathsf{C}_1 + c_2\mathsf{C}_2 + c_3\mathsf{C}_3,$$

where the cofactors have the same values as before; thus

$$\mathsf{A}_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad \mathsf{A}_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \text{ and so on.}$$

If the elements of any row (or column) are multiplied in order by the cofactors of the corresponding elements of another row (or column), the sum of the products is zero.

Using the same notation as before, suppose that the sum is $a_2\mathsf{A}_3 + b_2\mathsf{B}_3 + c_2\mathsf{C}_3$.

This equals $+a_2 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

$$\therefore\ a_2A_3+b_2B_3+c_2C_3=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0,$$

since two rows are identical.

If each element of any row (or column) is expressed as the sum of two numbers, the determinant can be expressed as the sum of two determinants whose remaining rows (or columns) are unaltered.

Consider the determinant, $\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1+z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Denote the cofactors of the elements of the first row by A_1, B_1, C_1.

Then the determinant $=(a_1+x_1)A_1+(b_1+y_1)B_1+(c_1+z_1)C_1$

$$=(a_1A_1+b_1B_1+c_1C_1)+(x_1A_1+y_1B_1+z_1C_1)$$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any rows (or columns).

This means that, for example,

$$\Delta\equiv\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Denote the cofactors of a_1, b_1, c_1 in Δ by A_1, B_1, C_1.

Then $\begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$=(a_1+pa_2+qa_3)A_1+(b_1+pb_2+qb_3)B_1+(c_1+pc_2+qc_3)C_1$$

$$=(a_1A+b_1B_1+c_1C_1)+p(a_2A_1+b_2B_1+c_2C_1)+q(a_3A_1+b_3B_1+c_3C_1)$$

$$=\Delta+0+0=\Delta.$$

This property is of great assistance in evaluating determinants; p and q may have any *positive or negative* values. The reader may

find it easier to understand the argument if the composite determinant is written out in full, thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

where each of the last two determinants, having two rows identical, is zero.

Example 4. Evaluate $\Delta \equiv \begin{vmatrix} 1 & 1 & 1 \\ 10 & 6 & 9 \\ 12 & 7 & 11 \end{vmatrix}$.

$$\Delta \equiv \begin{vmatrix} 1-1 & 1 & 1 \\ 10-6 & 6 & 9 \\ 12-7 & 7 & 11 \end{vmatrix} = \begin{vmatrix} 0 & 1-1 & 1 \\ 4 & 6-9 & 9 \\ 5 & 7-11 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 4 & -3 & 9 \\ 5 & -4 & 11 \end{vmatrix} = 4(-4)-(-3)5 = -1.$$

Example 5. Evaluate $\Delta \equiv \begin{vmatrix} 16 & 12 & 17 \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix}$.

$$\Delta = \begin{vmatrix} 16-(3\times 5) & 12-(3\times 4) & 17-(3\times 6) \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1+1 \\ 5 & 4 & 6+5 \\ 9 & 7 & 11+9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 4 & 11 \\ 9 & 7 & 20 \end{vmatrix} = 4\times 20 - 11\times 7 = 3.$$

EXERCISE IX. b

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$. 2. $\begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix}$. 3. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix}$.

4. $\begin{vmatrix} 1 & 4 & 5 \\ 1 & 6 & 6 \\ 1 & 9 & 7 \end{vmatrix}$. 5. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{vmatrix}$. 6. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$.

7. $\begin{vmatrix} 20 & 50 & 30 \\ 18 & 45 & 27 \\ 15 & 7 & 13 \end{vmatrix}$. 8. $\begin{vmatrix} 4 & 7 & 11 \\ 3 & 0 & 9 \\ 8 & 5 & 13 \end{vmatrix}$. 9. $\begin{vmatrix} 6 & 4 & 3 \\ 5 & 5 & 5 \\ 7 & 8 & 7 \end{vmatrix}$.

10. $\begin{vmatrix} 8 & 5 & 5 \\ 9 & 5 & 10 \\ 13 & 8 & 9 \end{vmatrix}$. 11. $\begin{vmatrix} 3 & 7 & 11 \\ 4 & 7 & 10 \\ 1 & 2 & 3 \end{vmatrix}$. 12. $\begin{vmatrix} 7 & 2 & 3 \\ 14 & 5 & 6 \\ 7 & 3 & 3 \end{vmatrix}$.

13. $\begin{vmatrix} 29 & 38 & 40 \\ 24 & 32 & 34 \\ 19 & 26 & 28 \end{vmatrix}$. 14. $\begin{vmatrix} 6 & 5 & 9 \\ 10 & 7 & 15 \\ 8 & 11 & 12 \end{vmatrix}$. 15. $\begin{vmatrix} 14 & 13 & 7 \\ 8 & 7 & 4 \\ 5 & 6 & 3 \end{vmatrix}$.

16. $\begin{vmatrix} 21 & 10 & 25 \\ 26 & 13 & 32 \\ 15 & 7 & 17 \end{vmatrix}$. 17. $\begin{vmatrix} 46 & 27 & 18 \\ 38 & 21 & 17 \\ 16 & 9 & 7 \end{vmatrix}$. 18. $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$.

Expand the following determinants :

19. $\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$. 20. $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$. 21. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

22. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. 23. $\begin{vmatrix} 1 & a & -b \\ -a & 1 & c \\ b & -c & 1 \end{vmatrix}$. 24. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

Eliminate x, y, z from the equations :

25. $x+by+cz=0,\quad ax+y+cz=0,\quad ax+by+z=0.$

26. $ax+y-cz=0,\quad bx-cy-z=0,\quad x-ay+bz=0.$

27. $a=\dfrac{x}{y-z},\quad b=\dfrac{y}{z-x},\quad c=\dfrac{z}{x-y}.$

28. Expand $\begin{vmatrix} x^2+y^2+1 & x^2+2y^2+3 & x^2+3y^2+4 \\ y^2+2 & 2y^2+6 & 3y^2+8 \\ y^2+1 & 2y^2+3 & 3y^2+4 \end{vmatrix}$.

29. Prove that the equation $\begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 5-\lambda & 6 \\ 3 & 4 & 10-\lambda \end{vmatrix}=0$ has three roots, and find them.

30. Write $\begin{vmatrix} ap+bq & ar+bs \\ cp+dq & cr+ds \end{vmatrix}$ as the sum of four determinants and hence prove that it equals $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} p & q \\ r & s \end{vmatrix}$.

31. Simplify $\begin{vmatrix} 1 & bc & bc^2+b^2c \\ 1 & ca & ca^2+c^2a \\ 1 & ab & ab^2+a^2b \end{vmatrix}$.

Factors

It is often possible to factorise a determinant by using the remainder theorem.

Example 6. Factorise $\Delta \equiv \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$.

If $a=b$, the first and second columns are identical and therefore $\Delta=0$.

$\therefore$ $(a-b)$ is a factor of Δ; similarly, $(b-c)$ and $(c-a)$ are factors of Δ.

But Δ is of the 4th degree in a, b, c, and is unaltered when a, b, c are replaced by b, c, a or by c, a, b; therefore the remaining factor must be of the form $k(a+b+c)$ where k is a constant (independent of a, b, c).

$$\therefore \quad \Delta = k(a-b)(b-c)(c-a)(a+b+c).$$

But the leading term of Δ is $+bc^3$, therefore, equating coefficients, we have $1=k$.

Equations

The simultaneous linear equations,

$$a_1x+b_1y+c_1z=0, \quad a_2x+b_2y+c_2z=0, \quad a_3x+b_3y+c_3z=0$$

are always satisfied by $x=y=z=0$. If they have any other solution, say one in which $z \neq 0$, they can be written

$$a_1\left(\frac{x}{z}\right)+b_1\left(\frac{y}{z}\right)+c_1=0, \text{ etc.}$$

The condition that they have a common solution (other than $x=y=z=0$) is by p. 172,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

It may be noted that in analytical geometry this gives the condition that the three straight lines represented by these equations are concurrent. Special cases, such as $a_2b_3 - b_2a_3 = 0$, correspond in Cartesian coordinates to parallelism.

Example 7. Solve the simultaneous equations,

$$3x + 2y + 4z = 19,\ \ 2x - y + z = 3,\ \ 6x + 7y - z = 17.$$

The equations may be written

$$\begin{aligned} 3x + 2y + (4z - 19)t &= 0, \\ 2x - y + (z - 3)t &= 0, \\ 6x + 7y - (z + 17)t &= 0, \end{aligned}$$

where $t = 1$.

Since $x = y = t = 0$ is *not* true, it follows from the above that the equations have a common solution, *only* if

$$\begin{vmatrix} 3 & 2 & 4z - 19 \\ 2 & -1 & z - 3 \\ 6 & 7 & -z - 17 \end{vmatrix} = 0.$$

$$\therefore\ z \begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 6 & 7 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 19 \\ 2 & -1 & 3 \\ 6 & 7 & 17 \end{vmatrix}.$$

Hence $\qquad 78z = 234, \quad \therefore\ z = 3.$

Similarly, we may find the values of x and y.

The method of Example 7 can be applied to the general case. Consider the equations,

$$a_1x + b_1y + c_1z + d_1 = 0, \quad \text{.........................(i)}$$
$$a_2x + b_2y + c_2z + d_2 = 0, \quad \text{.........................(ii)}$$
$$a_3x + b_3y + c_3z + d_3 = 0. \quad \text{.......................(iii)}$$

Proceeding as in Example 7, we have

$$\begin{vmatrix} a_1 & b_1 & c_1z + d_1 \\ a_2 & b_2 & c_2z + d_2 \\ a_3 & b_3 & c_3z + d_3 \end{vmatrix} = 0;$$

$$\therefore\ z \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Similarly,

$$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

$$\text{and } y\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = +\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}.$$

If $\Delta \neq 0$, these results can be written in the form,

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

It is, however, better in the general case to proceed as below, although in numerical work, especially if only one unknown is required, the method of Example 7 is convenient.

Let A_1, A_2, A_3 be the cofactors of a_1, a_2, a_2 in

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then, from p. 175, $a_1A_1 + a_2A_2 + a_3A_3 = \Delta$,

also $\qquad b_1A_1 + b_2A_2 + b_3A_3 = 0,\; c_1A_1 + c_2A_2 + c_3A_3 = 0.$

Multiply equations (i), (ii), (iii) on p. 180 by A_1, A_2, A_3, and add;

then $\qquad \Delta x + (d_1A_1 + d_2A_2 + d_3A_3) = 0.$

Similarly, $\qquad \Delta y + (d_1B_1 + d_2B_2 + d_3B_3) = 0,$

and $\qquad \Delta z + (d_1C_1 + d_2C_2 + d_3C_3) = 0,$

which give the same results as before, provided $\Delta \neq 0$.

If $\Delta = 0$, the equations are inconsistent unless

$$d_1A_1 + d_2A_2 + d_3A_3 = 0, \text{ etc.,}$$

and then the equations are not usually independent.

In analytical solid geometry, these equations represent planes, and the exceptional cases $(\Delta = 0)$ arise when the planes have a common line and when they are parallel to the same line or to one another.

The Product of two determinants of the second order was expressed as a second order determinant in Ex. IX b, No. 30. Similarly the product of two determinants of the third order can be expressed as a determinant of the third order.

If $\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' \equiv \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$,

and if

$$\mathsf{D} \equiv \begin{vmatrix} a_1x_1+b_1x_2+c_1x_3 & a_1y_1+b_1y_2+c_1y_3 & a_1z_1+b_1z_2+c_1z_3 \\ a_2x_1+b_2x_2+c_2x_3 & a_2y_1+b_2y_2+c_2y_3 & a_2z_1+b_2z_2+c_2z_3 \\ a_3x_1+b_3x_2+c_3x_3 & a_3y_1+b_3y_2+c_3y_3 & a_3z_1+b_3z_2+c_3z_3 \end{vmatrix},$$

then $$\Delta \times \Delta' = \mathsf{D}.$$

The determinant D can be expressed as the sum of $3 \times 3 \times 3$ determinants. One of these is

$$\begin{vmatrix} c_1x_3 & a_1y_1 & b_1z_2 \\ c_2x_3 & a_2y_1 & b_2z_2 \\ c_3x_3 & a_3y_1 & b_3z_2 \end{vmatrix} = x_3y_1z_2 \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix} = x_3y_1z_2\Delta.$$

Another of these is

$$\begin{vmatrix} c_1x_3 & a_1y_1 & c_1z_3 \\ c_2x_3 & a_2y_1 & c_2z_3 \\ c_3x_3 & a_3y_1 & c_3z_3 \end{vmatrix} = x_3y_1z_3 \begin{vmatrix} c_1 & a_1 & c_1 \\ c_2 & a_2 & c_2 \\ c_3 & a_3 & c_3 \end{vmatrix} = 0.$$

Of the 27 determinants, arranged in this way, all are zero except the 6 which contain all the letters a, b, c, and each of these has Δ as a factor; $\therefore$ Δ is a factor of D, and it is easy to see that the quotient $\mathsf{D} \div \Delta$ equals Δ'.

Alternatively, interchanging rows and columns, it follows that Δ' is a factor of D.

But $\Delta \times \Delta'$ and D are each of degree 6, and the term

$$a_1b_2c_3 \,.\, x_1y_2z_3$$

is a term in each ; $\therefore$ $\Delta \times \Delta' = \mathsf{D}$.

Example 8. If A_1, B_1, ... are the cofactors of a_1, b_1, ... in

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

prove that

$$\text{(i)}\quad \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1\Delta\,; \qquad \text{(ii)}\quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2,$$

(i) By the formula for a product,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & A_2 & A_3 \\ 0 & B_2 & B_3 \\ 0 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+0+0 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2+0+0 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3+0+0 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & \Delta & 0 \\ a_3 & 0 & \Delta \end{vmatrix} = a_1\Delta^2, \text{ by the relations on p. 175 ;}$$

$$\therefore\ \Delta(B_2C_3 - C_2B_3) = a_1\Delta^2\,; \qquad \therefore\ B_2C_3 - C_2B_3 = a_1\Delta.$$

(ii) As in (i),

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3\,;$$

$$\therefore\ \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \div \Delta = \Delta^2.$$

Determinants of any order may be defined by the use of minors. Thus the arrangement

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is called a *determinant of the fourth order*, and its value is, by definition,

$$+a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

where the coefficients of a_1, b_1, c_1, d_1 are obtained by deleting from the given determinant the row and column which pass

through the corresponding element and are called the **minors** of a_1, b_1, c_1, d_1.

The determinant can also be expanded with reference to the other rows, in the forms

$$-a_2\alpha_2+b_2\beta_2-c_2\gamma_2+d_2\delta_2\,;\quad +a_3\alpha_3-b_3\beta_3+c_3\gamma_3-d_3\delta_3\,;\ \text{etc.},$$

or with reference to the columns, in the forms

$$+a_1\alpha_1-a_2\alpha_2+a_3\alpha_3-a_4\alpha_4\,;\quad -b_1\beta_1+b_2\beta_2-b_3\beta_3+b_4\beta_4\,;\ \text{etc.},$$

where $\alpha_1, \beta_1, \ldots$ denote the *minors* of $a_1, b_1, \ldots$.

The expansion of a determinant of the fourth order contains $4!$ terms $=24$ terms. Similarly, we can proceed to determinants of the fifth order, containing $5!$ terms, and so on to any order.

The treatment of third order determinants can be applied to determinants of any order, and it then follows that the fundamental theorems established above for third order determinants also hold in the general case.

EXERCISE IX. c

Express in factors the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$. **2.** $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$.

3. $\begin{vmatrix} a & a & b+c \\ a+c & b & b \\ c & a+b & c \end{vmatrix}$. **4.** $\begin{vmatrix} 1 & x-y & (x-y)^2 \\ 1 & y-z & (y-z)^2 \\ 1 & z-x & (z-x)^2 \end{vmatrix}$.

5. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. **6.** $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$.

7. $\begin{vmatrix} a^2 & (b+c)^2 & bc \\ b^2 & (c+a)^2 & ca \\ c^2 & (a+b)^2 & ab \end{vmatrix}$. **8.** $\begin{vmatrix} a & a-c & a-b \\ b & c+a & b-a \\ c & c-a & a+b \end{vmatrix}$.

Solve the equations:

9. $x+y+z=2,\quad x+2y+3z=1,\quad 3x+y-5z=4.$

10. $2x-y-z=6,\quad x+3y+2z=1,\quad 3x-y-5z=1.$

11. $x+y+z=4,\quad x+2y+3z=9,\quad 3x+y+4z=12.$

12. $x+y-z=1,\quad 2x+y+z=7,\quad x-5y+3z=-3.$

13. $x+2y-3z=0, \quad 3x+3y-z=5, \quad x-2y+2z=1.$

14. $x+2y-z=5, \quad 3x-y+3z=7, \quad 2x+3y+z=11.$

15. Express the square of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as a second order determinant.

16. If $A_1, B_1, C_1, \ldots$ are the cofactors of $a_1, b_1, c_1, \ldots$ in

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

prove that $A_2B_3 - B_2A_3 = c_1\Delta$, and find a similar expression for $A_2C_3 - C_2A_3$.

17. Express the square of $\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}$ as a third order determinant, without any zero elements. What is its value?

18. Find the value of $\begin{vmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix}$, and express its square as a third order determinant.

19. Prove that $\begin{vmatrix} a & b & c & d \\ a & -b & -c & -d \\ a & b & -c & -d \\ a & b & c & -d \end{vmatrix} = -8abcd.$

20. Prove that $\begin{vmatrix} 1 & a & a & a \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix} = (b-a)^3.$

21. Prove that $\begin{vmatrix} a & b & c & d \\ b & a & c & d \\ a & b & d & c \\ b & a & d & c \end{vmatrix} = 0.$

22. Prove that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$

Factorise :

23. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}.$

24. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$

25. Prove that the result of eliminating x from the equations, $a_1x^2+b_1x+c_1=0$, $a_2x^2+b_2x+c_2=0$, may be written

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ 0 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_2 & b_2 & c_2 \end{vmatrix}=0.$$

TEST PAPERS A. 21-35

A. 21

1. (i) Eliminate a, b from the equations,

$$a^m=b^h a^n=b^k a^p.$$

(ii) Express by factorials and indices the product

$$(1-\tfrac{1}{2})(1-\tfrac{1}{4})(1-\tfrac{1}{6})\ldots\left(1-\frac{1}{2n}\right).$$

2. Find a quadratic equation such that the arithmetic mean and the harmonic mean between its roots are respectively $\frac{7}{3}$ and $\frac{5}{7}$.

3. (i) Find the coefficient of x^n in the expansion of $\dfrac{1+x+4x^2}{(1-x)^3}$ in ascending powers of x.

(ii) Use the exponential series to evaluate e^{-x} to 3 places of decimals if $x=0{\cdot}4$.

4. Prove that $\dfrac{(x-5)(x-1)}{2x-1}$ can assume all values except those between -4 and -1. Illustrate by a rough graph.

5. What general statements can be made about the roots of the following equations:

(i) $x^3+qx+r=0$; (ii) $x^3+px^2+px+q=0$;
(iii) $ax^3+bx^2+bx+a=0$; (iv) $x^4+qx^2+r=0$?

A. 22

1. (i) Find k if $(3x-2y)^2+k(x-y)(x-2y)$ is a perfect square.

(ii) Find the least value of x^2-4x-7.

2. (i) Sum to n terms,

$$(a+1)(a-1)+(a+2)(a-2)+(a+3)(a-3)+\ldots.$$

(ii) Sum to infinity, if $x>1$,

$$1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{1.2}\left(1-\frac{1}{x}\right)^2+\frac{n(n+1)(n+2)}{1.2.3}\left(1-\frac{1}{x}\right)^3+\ldots.$$

3. Eliminate x, y from the equations,

$$x^2+xy=a^2,\quad y^2+xy=b^2,\quad x^2+y^2=c^2.$$

4. Prove by induction that the sum of n terms of

$$1+\frac{2n-2}{2n-3}+\frac{(2n-2)(2n-4)}{(2n-3)(2n-5)}+\dots \text{ is } 2n-1.$$

5. (i) If x is small, find an approximation for

$$\frac{2n\log(1+x)}{(1+x)^n-(1+x)^{-n}}, \text{ neglecting } x^3.$$

(ii) Prove that $e^x=2x+1$ has a root between 1·2 and 1·3.

A. 23

1. (i) Find the range of values of x if $5x-1<(x+1)^2<7x-3$.
   (ii) Find the conditions that the roots of $x^2+bx+c=0$ are respectively $\frac{3}{4}$ of those of $x^2+qx+r=0$.

2. Express in partial fractions $\dfrac{x^2+3}{(x+1)(x^2+1)}$.

Find the coefficients of x^{2r} and x^{2r+1}, if this function can be expanded in ascending powers of x.

3. (i) Sum to n terms, $1.2+3.4+5.6+\dots$.
   (ii) Evaluate $\sqrt[4]{1{\cdot}03}$ correct to 6 places of decimals.

4. If x is real, prove that $\dfrac{x+1}{x^2-4}$ is capable of any value, and illustrate by a rough graph.

5. (i) If $x>1$ and if $y=\dfrac{1}{2x^2-1}$, prove that the sum to infinity of $y+\frac{1}{3}y^3+\frac{1}{5}y^5+\dots$ is $\log x-\log\sqrt{(x^2-1)}$.
   (ii) Sum to infinity, $\dfrac{1^2}{2!}+\dfrac{2^2}{3!}+\dfrac{3^2}{4!}+\dots$.

A. 24

1. (i) Express $\dfrac{1+\sqrt{3}}{(2+\sqrt{3})(3\sqrt{3}+5)}$ in the form $a+b\sqrt{3}$, where a, b are rational.
   (ii) If $x^2+y^2=7xy$, prove that
   $$\log(x+y)=\log 3+\tfrac{1}{2}\log x+\tfrac{1}{2}\log y.$$

2. (i) Prove that $(x-1)^2$ is a factor of $x^n-nx+n-1$.
   (ii) What is the condition that $x^2-3xy-10y^2$ and $ax^2+2hxy+by^2$ have a common factor?

3. (i) Prove that ${}_{n+2}\mathrm{P}_r={}_n\mathrm{P}_r+2r\,.\,{}_n\mathrm{P}_{r-1}+r(r-1)\,.\,{}_n\mathrm{P}_{r-2}$.
   (ii) If n is a positive integer, prove that the sum of
   $$a-(a+b)n+(a+2b)\frac{n(n-1)}{1.2}-(a+3b)\frac{n(n-1)(n-2)}{1.2.3}+\dots$$
   to $(n+1)$ terms is zero.

4. Find which is the numerically greatest term in the expansion of $(1+5x)^{-4}$ when $x=\frac{1}{7}$.

5. Find the coefficient of x^n in the expansions of
(i) $(1-x)^2e^x$, (ii) $\log(1-2x+x^2)$, in ascending powers of x.

A. 25

1. If $x+y+z=0$, prove that
(i) $\Sigma(x^2y)=-3xyz$; (ii) $\Sigma(x^3)=3xyz$.

2. (i) If α, β are the roots of $x^2+qx+r=0$, form the equation whose roots are $\alpha^2-\alpha\beta$, $\beta^2-\alpha\beta$.
(ii) If one root of $x^3+px^2+qx+r=0$ equals the sum of the other two, prove that $p^3+8r=4pq$.

3. Find the limits within which the values of $\dfrac{x+4}{(x+1)(x-8)}$ cannot lie, and illustrate by a rough graph.

4. (i) Find the sum to n terms of
$$1+2(1-a)+3(1-a)^2+4(1-a)^3+\dots.$$
(ii) Find the sum to infinity of the series,
$$\frac{2}{9}+\frac{2}{2!}\left(\frac{2}{9}\right)^2+\frac{2.5}{3!}\left(\frac{2}{9}\right)^3+\frac{2.5.8}{4!}\left(\frac{2}{9}\right)^4+\dots.$$

5. (i) If x is small, prove that
$$\left(\frac{1-x}{1+x+x^2}\right)^n \simeq 1-2nx+n(2n-1)x^2.$$
(ii) Expand $\log(e^x+e^{-x})$ in powers of x as far as x^4.

A. 26

1. (i) For what values of x is $\dfrac{x^2+2x-19}{x-4}$ greater than 4 ?
(ii) If $x+y+z=0$,
express $ayz+bzx+cxy$ in the form $px^2+qy^2+rz^2$.

2. Express in partial fractions, $\dfrac{(3x-1)(x+2)}{(x+1)(x^3-1)}$.

3. (i) Sum to n terms, $1.2^2+2.3^2+3.4^2+\dots$.
(ii) Sum to n terms and to infinity, $\dfrac{1}{1.4}+\dfrac{1}{4.7}+\dfrac{1}{7.10}+\dots$.

4. (i) Find between what integers the roots of the equation $x^3-27x-36=0$ lie.
(ii) If $b^2>ac$, prove that
$$(a+c)(ax^2+2bx+c)-2(ac-b^2)(x^2+1)$$
has the same sign for all values of x.

5. Sum to infinity, $\dfrac{3}{2.4}-\dfrac{4}{2.4.6}+\dfrac{5}{2.4.6.8}-\dots$.

A. 27

1. (i) Eliminate x, y from the equations,

$$xy+x=a, \quad xy+y=b, \quad x+y=c.$$

(ii) Find the condition, independent of k, that the roots of

$$ax^2+bx+c=0 \text{ are } \frac{k+1}{k} \text{ and } \frac{k+2}{k+1}.$$

2. Find the coefficient of x^n in the expansion of $\frac{1}{1-x+x^2-x^3}$ in ascending powers of x.

3. (i) Sum to n terms $1+4x+7x^2+10x^3+\ldots$ by multiplying the expression by $(1-x)^2$.

(ii) Sum to infinity, $\frac{1}{2!\,4}+\frac{2}{3!\,5}+\frac{3}{4!\,6}+\ldots$.

4. (i) Solve $27x^3-36x-16=0$, given that it has two equal roots.

(ii) Prove that $3x^5-5qx^3+3r=0$ has 3 real roots if $4q^5>9r^2$.

5. Factorise $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$.

A. 28

1. (i) If x_1, x_2 are the roots of $ax^2+bx+c+\lambda(px^2+qx+r)=0$, prove that $x_1x_2(aq-bp)-(x_1+x_2)(cp-ra)+br-qc=0$.

(ii) Factorise $(x^3+y^3)(x-y)+(y^3+z^3)(y-z)+(z^3+x^3)(z-x)$.

2. In how many ways can 4 boys and 4 girls sit in a row if no two boys sit next to one another?

3. Prove that $\frac{2x^2-14x+11}{2x^2-2x+5}$ lies between -1 and 3.

4. (i) Sum to n terms and to infinity,

$$\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots.$$

(ii) Sum to infinity,

$$\frac{1}{2.1}+\frac{1}{3.1!}+\frac{1}{4.2!}+\frac{1}{5.3!}+\ldots.$$

5. (i) Solve the equations,

$$x+y+z=-2, \; x^2+y^2+z^2=66, \; (y+z)(z+x)(x+y)=90.$$

(ii) Find the real root of $x^3+6x=2$ to 4 places of decimals.

A. 29

1. (i) If α, β are the roots of $a(x^2-1)+2bx=0$, prove that the equation whose roots are $2\alpha+\dfrac{1}{\beta}$, $2\beta+\dfrac{1}{\alpha}$ is the same.

(ii) Factorise $(x+y+z)(yz+zx+xy)-xyz$.

2. Find the coefficient of x^n in the expansion of $\dfrac{2x^3}{(1+x^2)(1-x)^2}$ in ascending powers of x if n is (i) odd, (ii) even.

3. Sum to infinity, (i) $\dfrac{1}{6}+\dfrac{1.4}{6.12}+\dfrac{1.4.7}{6.12.18}+\dots$.

(ii) $1-\dfrac{2^3}{1!}+\dfrac{3^3}{2!}-\dfrac{4^3}{3!}+\dots$.

4. (i) If x is small, prove that

$$\left\{\sqrt{\left(4+\frac{4x}{3}\right)}-\sqrt[3]{(8+4x)}\right\}\div\sqrt[3]{(1+\tfrac{1}{3}x)}\simeq\frac{x^2}{36}-\frac{x^3}{72}.$$

(ii) Find an approximation for

$x\sqrt{(1+x)}+\log(1-x)$, neglecting x^5.

5. Prove that $\begin{vmatrix} a^2 & a^2 & (b+c)^2 \\ (a+c)^2 & b^2 & b^2 \\ c^2 & (a+b)^2 & c^2 \end{vmatrix}=2abc(a+b+c)^3$.

A. 30

1. (i) If $ax^2+bx+1\frac{1}{2}c=0$ has real roots of the same sign, prove that $a^2x^2+(4ac-b^2)x+c^2=0$ has positive roots.

(ii) Find the equation whose roots are each less by 2 than the roots of $x^3-6x^2+7=0$.

2. Find the number of permutations, 5 at a time, of 7 things of which 3 are alike and the rest all different.

3. (i) Prove by induction that

$$2+\frac{2}{3}+\frac{2.4}{3.5}+\dots+\frac{2.4.6\dots(2n)}{3.5.7\dots(2n+1)}=\frac{2.4.6\dots(2n+2)}{3.5.7\dots(2n+1)}.$$

(ii) Sum to infinity,

$$\frac{5}{3.6}+\frac{5.7}{3.6.9}+\frac{5.7.9}{3.6.9.12}+\dots.$$

4. (i) Find the coefficient of x^{3n+1} in $\dfrac{1+x}{(1+x+x^2)^2}$.

(ii) Express in partial fractions, $\dfrac{2a(a+x)}{(a-x)^2(a^2+x^2)}$.

5. Eliminate x, y, z from the equations,

$x^3+y^3+z^3=dxyz$, $bcx+cay+abz=0$, $a^2x+b^2y+c^2z=0$.

A. 31

1. If x, y, z are positive and unequal, prove that

(i) $x+y-2\sqrt{(xy)}>0$, (ii) $(x+y)(y+z)(z+x)>8xyz$.

2. What is the condition that $\dfrac{3+x}{(2-x)(1+x)}$ may be expanded in the form $a_0+a_1x+a_2x^2+\dots$?

And in this case, sum to infinity $a_0^2+a_1^2x+a_2^2x^2+\dots$.

3. (i) If α, β, γ are the roots of $x^3+qx+r=0$, prove that $\beta^3+\gamma^3$, $\gamma^3+\alpha^3$, $\alpha^3+\beta^3$ are the roots of

$$(x+2r)^3+q^3(x+3r)=0.$$

(ii) Solve $4x^3+4x^2-7x+2=0$, given that two of the roots are equal.

4. Sum to infinity:

(i) $\dfrac{1^3}{1!}+\dfrac{1^3+2^3}{2!}+\dfrac{1^3+2^3+3^3}{3!}+\dots$.

(ii) $\dfrac{1}{1.2.3.4}+\dfrac{1}{3.4.5.6}+\dfrac{1}{5.6.7.8}+\dots$.

5. Find x in terms of a, b, c, given $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix}=0.$

A. 32

1. (i) Find p, q if x^2-x-2 is a factor of $x^6-2x^5+px^3-qx+2$.

(ii) Prove that $2(b-c)^4+2(c-a)^4+2(a-b)^4$ is a perfect square.

2. Prove that $x=k(x-1)(x+2)$ has real roots for all real values of k, but that $x=k(x-1)(x-2)$ has real roots only if k does not lie between $-3-2\sqrt{2}$ and $-3+2\sqrt{2}$.

Draw rough graphs of $\dfrac{x}{(x-1)(x+2)}$ and $\dfrac{x}{(x-1)(x-2)}$.

3. (i) Find between what integers the roots of the equation $x^3-3x^2-4x+11=0$ lie.

(ii) Find the positive root of $x^3=2x+5$ to 3 places of decimals.

4. (i) Sum to n terms and to infinity,

$$\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\dots.$$

(ii) Sum to infinity, $\dfrac{1.3.5}{2!}+\dfrac{3.5.7}{4!}+\dfrac{5.7.9}{6!}+\dots$.

5. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ is a perfect cube.

A. 33

1. (i) Find the value of λ if the roots α, β of $5x^2+(2\lambda+1)x+\lambda-2=0$ are such that $2\alpha+5\beta=1$.

(ii) Solve $\dfrac{(1-x)^2}{2-x^2}=\dfrac{(1-a)^2}{2-a^2}$ and prove that a cannot be equal to 1 or 2 if $x \neq a$.

2. Express in partial fractions $\dfrac{16(x-2)(x-4)(x-6)}{(x-1)(x-3)(x-5)(x-7)}$.

3. Prove that for real values of a, x,

$$0 \leqslant \frac{(x+a)^2}{x^2+x+1} \leqslant \frac{4}{3}(a^2-a+1).$$

Sketch the graph of $\dfrac{(x-1)^2}{x^2+x+1}$.

4. If p is a positive integer, prove that the number of positive integral solutions (including zero solutions) of $2(x-p)+y+z=0$ is $(p+1)^2$.

5. (i) If $s_r=1^r+2^r+3^r+\ldots+n^r$, prove that $s_5+s_7=2s_1{}^4$.

(ii) Sum to infinity, $\dfrac{2}{1.2.3}+\dfrac{3}{3.4.5}+\dfrac{4}{5.6.7}+\ldots$.

A. 34

1. If $\lambda>1$, prove that both roots of $(1-\lambda)x^2+3\lambda x-1=0$ are positive and that one of them is greater than 3.

2. (i) If $\dfrac{1}{(1-x)(1-x^2)(1-x^4)}$ is expanded in ascending powers of x, prove that the coefficient of x^{4n} is $(n+1)^2$, n being integral.

(ii) If $(1-x)^{-3}=1+c_1x+c_2x^2+\ldots$, find the value of

$$1+c_1+c_2+\ldots+c_n.$$

3. (i) If α, β, γ are the roots of $x^3+qx+r=0$, form the equation whose roots are $\alpha^2(\beta+\gamma)$, $\beta^2(\gamma+\alpha)$, $\gamma^2(\alpha+\beta)$.

(ii) Prove that $x^4+x^3-4x^2+4x+1=0$ has two roots between 1 and 2, and find the positions of the other two roots.

4. Find the greatest and least values of $\dfrac{x^2-2x+2}{x^2+3x+9}$.

5. Factorise $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$.

A. 35

1. (i) If $ax^2+2hxy+by^2$ is transformed into $\mathrm{AX}^2+2\mathrm{HXY}+\mathrm{BY}^2$ by the substitutions, $x\sqrt{2}=\mathrm{X}+\mathrm{Y}$, $y\sqrt{2}=\mathrm{X}-\mathrm{Y}$, prove that (i) $\mathrm{A}+\mathrm{B}=a+b$, (ii) $\mathrm{AB}-\mathrm{H}^2=ab-h^2$.

   (ii) If $\dfrac{a^2-d^2}{b^2c^2-d^4}=\dfrac{a}{bcd}$, prove that either $\dfrac{a}{b}=\dfrac{c}{d}$ or $d^3=-abc$.

2. (i) Factorise $(a+b+c)^3-\Sigma(b+c-a)^3$.

   (ii) Express $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$ as the sum of three squares. What follows if this expression is zero?

3. (i) Sum to n terms, $1.2.5+3.4.7+5.6.9+\dots$.

   (ii) Sum to infinity, $\dfrac{1}{1.2.3}+\dfrac{1}{2.3.5}+\dfrac{1}{3.4.7}+\dfrac{1}{4.5.9}+\dots$.

4. If $p>q>0$, prove that $x^3+x^2=px+q$ has three real roots.

5. If $|x|<1$, prove that the sum to infinity of

$$\frac{2^2x^2}{1.4}+\frac{3^2x^3}{2.5}+\frac{4^2x^4}{3.6}+\dots \text{ is } \frac{4-x^3}{3x^2}\log(1-x)+\frac{12-6x-2x^2+5x^3}{9x(1-x)}.$$

ANSWERS

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Exercise I. *a*. (p. 4.)

1. 54. 2. 120. 3. 1000. 4. 90. 5. 720. **6.** 480.

7. 26^4. 8. $10!$; 5040. 9. 720 ; 210 ; 56.

10. $\frac{10!}{7!}$; $\frac{13!}{9!}$; $\frac{n!}{(n-3)!}$; $\frac{(n+3)!}{(n-1)!}$; $\frac{(n+3)!}{(n-4)!}$; $\frac{(n+r)!}{(n-1)!}$.

11. 24. 12. 43. 13. 5040 ; 720 ; 120. **14.** 40320 ; 1440.

15. 48. 16. 4, 18. **17.** n ; $(n-1)\{(n-1)!\}$.

18. 720 ; 600 ; 96. 19. $n(n-1)$.

Exercise I. *b*. (p. 8.)

1. 30 ; 720. 2. $\frac{10!}{4!}$. 3. 210. 4. 42. 5. 240.

6. 2520 ; 1800. 7. 20 ; 120 ; 120. 8. $n(n-1)$; $\frac{n!}{(n-5)!}$; $\frac{(2n)!}{n!}$.

9. 192. 10. 72. 11. 56. 12. 15120 ; 3360.

13. 630. 14. 144. 15. 576 ; 1728. 16. 30240 ; 4320.

17. 453600 ; 40320. 18. 120 ; 80 ; 625. 19. 240 ; 480.

20. 103680. 21. 144. 22. 120. 23. $2(9!)$; $5(9!)$.

24. 5760. 25. 34560. 26. $\frac{91(17!)}{216}$. 27. $42(8!)$.

Exercise I. *c*. (p. 13.)

1. 35. 2. 364. 3. 22100. 4. 10. 5. 66 ; 72 ; 95.

6. 11760. 7. 126. 8. 210. 9. 120. 10. 100. 11. 56 ; 196.

12. $\frac{1}{6}n(n-1)(n-2)$; $\frac{1}{2}(n-1)(n-2)$. 13. $\frac{1}{2}n(n-1)$; $\frac{1}{2}n(n-3)$.

14. 198 ; 252. 15. 350. 16. 7800. 17. 10640.

18. 8 ; 11. 19. $\frac{1}{3}(n+1)$; $\frac{n+1}{r+1}$. 20. $\frac{(4n)!}{(n!)^4}$.

Exercise I. *d*. (p. 17.)

1. 60. 2. $\frac{20!}{(10!)^2}$. 3. 15. 4. 1023. 5. 418. 6. 39.

7. 8. 8. 344. 10. 105. 11. 21 ; 11. 12. 119.

13. $140(8!)(7!)$. **14.** 84; 72. **15.** 1024; 216. **16.** $\dfrac{(3n)!}{(2n)!\,n!}$.

19. $\dfrac{(2n)!}{2^n(n!)}$. **20.** 155925; 10800. **21.** $2^n(n+1)-1$.

22. $\frac{1}{2}mn(m+n-2)$; $\frac{1}{2}mn(m+n)$.

Exercise I. *e*. (p. 19.)

1. $7(8!)$. **2.** $2(n-3)(n-4)\{(n-3)!\}$. **3.** 4804. **4.** 9.

5. 14175. **6.** 111. **7.** 504. **9.** 945; 90. **10.** 3360.

11. 2772; 126. **12.** $\dfrac{70!}{66!\,2!}$. **13.** 60. **14.** $\dfrac{(m+n)!}{m!\,n!}$.

15. $(n-3)(n-4)\{(n-2)!\}$. **17.** 1260.

18. $\frac{1}{2}m(m-1)+n(2m+n-1)$. **19.** 2520.

20. $\frac{1}{4}pq(p-1)(q-1)$. **21.** 60. **22.** $\dfrac{(p+q-1)!}{(p-1)!}$.

23. $\dfrac{(n+r-1)\{(n-2)!\}}{r!(n-r-1)!}$. **25.** 2^n. **26.** $\dfrac{(2n)!}{n!\,n!}$.

27. (i) and (ii) $\dfrac{(p-1)!}{(p-q-1)!\,q!}$.

Exercise II. *a*. (p. 23.)

1. $x^3+x^2\Sigma(a)+x\Sigma(ab)+abc$; $x^3+2x^2\Sigma(a)+4x\Sigma(ab)+8abc$;
$x^4+x^3\Sigma(a)+x^2\Sigma(ab)+x\Sigma(abc)+abcd$.

2. $\Sigma(a_1a_2)$; 10; $10a^2$, $10a^3$.

3. ${}_nC_2$, ${}_nC_2$, ${}_nC_r$; ${}_nC_2a^2$, ${}_nC_2a^{n-2}$, ${}_nC_ra^{n-r}$.

4. $x^4+4x^3+6x^2+4x+1$. **5.** $y^4-4y^3+6y^2-4y+1$.

6. $x^3+6ax^2+12a^2x+8a^3$. **7.** $z^4+8z^3+24z^2+32z+16$.

8. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

9. $y^6+6y^4+15y^2+20+\dfrac{15}{y^2}+\dfrac{6}{y^4}+\dfrac{1}{y^6}$.

10. $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$.

11. $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

12. 9; fifth term; x^2; 56.

13. 21; $x^{15}y^5$, x^9y^{11}; $3^2.5^{19}.38x^{18}y^2$, $3^{18}.5^3.38x^2y^{18}$, $\dfrac{20!}{10!\,10!}(15xy)^{10}$.

14. 144, 189, 1215, 1750000. **15.** 1000, -20, $-120y^7$, 21.

16. $-70000a^4b^3$; ${}_nC_3(3x)^3$; ${}_nC_3x^{n-6}$.

17. ${}_nC_rx^{n-r}y^r$; ${}_nC_ra^{n-r}(-2b)^r$; ${}_nC_rx^{n-2r}$.

18. $2\{1+{}_nC_2x^2+{}_nC_4x^4+\ldots\}$; $2\{x+{}_nC_3x^3+{}_nC_5x^5+\ldots\}$.

19. $1{\cdot}025283$. **21.** $3^6.55$. **22.** ${}_{2n}C_n(\frac{1}{4})^n$.

23. ${}_{2n}C_n$; $(-1)^n {}_{2n+1}C_n x$, $(-1)^{n+1} {}_{2n+1}C_n \left(\frac{1}{x}\right)$.

24. a^3. **25.** x^4. **26.** -5 ; $\frac{n(n-1)(n-2)(n-7)}{24}$.

27. $1+2x-9x^3-18x^4$; 168. **28.** 82. **29.** 10084 ; 1.

30. $(x^2+1)^n$.

Exercise II. *b*. (p. 27.)

1. $-2^4.3^3.35x^4y^3$; $2^{11}.3^3.55x^{-6}$; $-7.3^{-5}x^{11}$;

$\frac{n!}{(r-1)!(n-r+1)!}x^{n-2r+2}$.

2. 1320 ; $2^{13}.27.13$; $3^9.16$; $(-1)^r \frac{(4r)!}{(3r)!\,r!}$;

$(-1)^{r+1}\frac{(n-1)!}{(r+1)!\,(n-r-2)!}$; $\frac{(4r)!}{(3r)!\,r!}$.

3. -63.128 ; -1760 ; $\frac{(2n)!}{n!\,n!}$. **4.** $\frac{1287}{8}$; $-\frac{99}{16}$; $2^{11}.3^5.35.13$; $-\frac{231}{8}$.

5. $1-14x+105x^2-532x^3$. **6.** 1. **7.** $-\frac{1}{2}n(n-1)(n-2)$.

8. $\frac{(n-r+1)x}{2r}$; $-\frac{3(n-r+1)}{rx^2}$; $\frac{3y(r+1)}{2rx}$.

9. $\frac{(n-r+2)(n-r+1)y^2}{r(r-1)x^2}$; $\frac{9(n-r+2)(n-r+1)y^2}{4r(r-1)x^2}$. **10.** 11.

12. 252 ; 462 ; $\frac{(4n+3)!}{(2n+2)!\,(2n+1)!}$. **13.** 792, 924.

14. $(n+1)$th term.

15. (i) 4th, 5th ; (ii) 9th ; (iii) 9th, 10th ; (iv) 4th ; (v) 4th.

18. 0. **19.** $x^7+x^5-3x^3-3x+3x^{-1}+3x^{-3}-x^{-5}-x^{-7}$.

20. $1+nx+\frac{1}{2}n(n+1)x^2+\frac{1}{6}n(n-1)(n+4)x^3$.

22. $\frac{n!(n-2r+1)}{r!(n-r+1)!}$; $\frac{(n+2)!}{r!(n-r+2)!}$.

25. $\frac{n!}{p!\,q!\,r!}$; 4, 6, 12, 24. **26.** $\frac{(2n)!}{(n-1)!(n+1)!}$.

27. $\frac{(n+1)!}{r!(n-r+1)!}(2^{n-r+1}-1)$. **28.** 2, 3.

Exercise II. *c*. (p. 32.)

1. $3^n = 1+2c_1+4c_2+\dots$; $(-1)^n = 1-2c_1+4c_2-\dots$.

2. 256 ; 729. **3.** 125 ; 63 ; 62.

4. $1-10x+55x^2-200x^3+530x^4$; 32, 7776, -3872.

6. ${}_{n+1}C_r = c_r + c_{r-1}$; ${}_{n+2}C_r = c_r + 2c_{r-1} + c_{r-2}$;

${}_{n+3}C_r = c_r + 3c_{r-1} + 3c_{r-2} + c_{r-3}$.

8. 0. 9. $\dfrac{1}{n+1}\{(1+x)^{n+1}-1\}$.

10. (i) and (ii) $\frac{1}{2}\{(1+x)^n+(1-x)^n\}$. 11. 0.

12. 0. 19. 243; 1,000,000; 81.

21. $x^p+c_1x^{p+q}+c_2x^{p+2q}+\ldots$; result of No. 20 with $a=p$, $d=q$ and x^q for x.

23. ${}_{4n}C_{2n}-{}_{2n}C_1\cdot{}_{4n-2}C_{2n}+{}_{2n}C_2\cdot{}_{4n-4}C_{2n}-\ldots=4^n$.

27. $\frac{1}{2}(3^n+1)$; $(n+1)3^n$.

Exercise III. *a*. (p. 36.)

1. 4, 3, 2, $8a^3+3$, $27x^3+3$. 2. 2, 0, 0, $\dfrac{1-3x+2x^2}{x^2}$, x^2-5x+6.

3. 10, 100, 1, 0·1, 10^{-x}. 4. $2x+h$; $-\dfrac{1}{x(x+h)}$.

5. $1.2+2.3+\ldots+n(n+1)$. 6. $n!+(n+1)!+\ldots+(2n)!$

7. $1+2x+3x^2+\ldots+nx^{n-1}$. 8. $1^2-2^2+\ldots-(2n)^2$.

9. $(n+1)^3-1$. 10. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\dfrac{1}{n}$; $\log(n!)$. 11. $\sum\limits_1^n r(r+2)$.

12. $\sum\limits_1^{2n}(-1)^{r-1}\dfrac{1}{r}$. 13. $\sum\limits_0^n\dfrac{1}{(n+r)^2}$. 14. $\sum\limits_0^{n-1}(-1)^r f(a+rh)$.

15. $\frac{1}{2}n(n+1)$. 16. $(n+1)(3n-1)$. 17. $\dfrac{a(1-r^n)}{1-r}$.

18. $\frac{1}{2}n\{2a+(n-1)d\}$. 19. $\log\dfrac{n+1}{m}$. 20. $f(n+1)-f(1)$.

21. $f(m-1)-f(n)$. 22. $\frac{1}{3}n(n+1)(n+2)$. 23. $\dfrac{m}{m+1}$.

24. $\frac{3}{4}n^2(n+1)(5n+1)$. 25. $\frac{1}{3}n(n^2-1)$; $\frac{1}{3}(n+1)(n+2)(n+3)$.

Exercise III. *b*. (p. 41.)

1. 20.27.41. 2. 44100. 3. 101.1650.

4. 71.81.275. 5. $\frac{4}{3}n(2n+1)(4n+1)$. 6. $\frac{2}{3}n(16n^2-1)$.

7. $n^2(2n^2-1)$. 8. $\frac{1}{4}n^2(3n+1)(5n+3)$.

9. $r(r+1)-4r$; $3r(r+1)-2r+1$; $r(r+1)+r-3$.

10. $a=1$, $b=-3$, $c=2$, $d=0$; $a=1$, $b=6$, $c=d=15$.

11. $\frac{1}{3}n(n+1)(n+5)$. 12. $\frac{1}{4}n(n+2)(n^2-1)$.

13. $\frac{1}{4}(n^2+3n)(n^2+3n+4)$. 14. $\frac{1}{2}n(n+1)^2(n+2)$.

15. $\frac{1}{30}n(n+1)(n+2)(6n^2+57n+137)$.

16. $\frac{1}{6}n(n+1)(2n+7)$. 17. $\frac{1}{3}n(4n^2+6n-1)$.

18. $\frac{1}{6}n(n+1)(2n+31)$. 19. $\frac{1}{12}n(n+1)(3n^2+11n+4)$.

20. $\frac{1}{12}n(n+1)(n+2)(3n+5)$. 21. $\frac{1}{12}n(n+1)(n+2)(3n+13)$.

22. $\frac{1}{2}n(n+1)(-1)^{n-1}$.
23. $\frac{1}{6}n(n-1)(n-2)$.
24. $\frac{1}{12}n(n+1)^2(n+2)$.
25. $\frac{1}{6}n(n+1)(n+2)$.
26. $\frac{1}{6}n(2n+1)(7n+1)$.
27. 1840.
31. $\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$.
32. $10r^4+2r^2$; $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

Exercise III. *c.* (p. 43.)

14. $2^{n+1}-n-2$.

Exercise III. *d.* (p. 45.)

1. $\{1-(n+1)x^n+nx^{n+1}\}\div(1-x)^2$.
2. $\{1-x-(2n+1)(-x)^n+(2n-1)(-x)^{n+1}\}\div(1+x)^2$.
3. $\frac{1}{4}\{15-(6n+5)3^{1-n}\}$.
4. $\frac{1}{16}\{9+(4n+3)(-3)^{1-n}\}$.
5. $2a+2d-\{a+(n+1)d\}\,2^{1-n}$.
6. $\{x^{n+1}-(n+1)x(-y)^n+n(-y)^{n+1}\}\div(x+y)^2$.
7. $\{2-(n+1)(n+2)x^n+2n(n+2)x^{n+1}-n(n+1)x^{n+2}\}\div2(1-x)^3$.
8. $\dfrac{2(2x)^n-2}{2x-1}-\dfrac{x^n-1}{x-1}$.
9. 0.
10. 0.
11. 0.
12. $\dfrac{2(2x)^n-2}{2x-1}-\dfrac{1+x-x^n(n+1)^2+x^{n+1}(2n^2+2n-1)-n^2x^{n+2}}{(1-x)^3}$.

Exercise III. *e.* (p. 46.)

1. 45.
2. $\dfrac{x^{2n+2}-x^{-2n}}{x^2-1}$.
4. $\frac{4}{3}(2^{2n}-1)-n(n+2)$.
5. $n(3n^2+6n+1)$.
6. $\frac{1}{3}n(4n^2+12n-1)$.
7. $\frac{1}{6}n(2n+1)(7n+1)$.
8. $\frac{1}{4}n^2(3n+1)(5n+3)$.
10. $\frac{1}{6}n(n+1)(n+2)$.
18. $\dfrac{ku_1^2(k^{2n}-1)}{k^2-1}$.
19. $\frac{1}{4}n(n+1)(n+4)(n+5)$.
24. Each $=f(x)$.
25. $\dfrac{n(3n+1)}{4(n+1)(n+2)}$.
26. $\frac{1}{2}n(n-1)$; n^2-n+1; n^3; $\frac{1}{4}n^2(n+1)^2$.
28. $\dfrac{n(n+3)}{4(n+1)(n+2)}$.
29. $\dfrac{1}{18}-\dfrac{1}{3(n+1)(n+2)(n+3)}$.
30. $24r^7-8r^3$; $\frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$.
31. $14r^6-2r^2$; $\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$.

Test Papers A. 1-10. (pp. 48-52.)

A. 1. 1. 41·9; 137·5.
2. $14x^2+37x+14=0$.
3. $\dfrac{(2n)!}{(n!)^2}+\dfrac{(2n)!}{(n+3)!\,(n-3)!}$; 495.
4. 13.1700; 4.13^3.
5. $\frac{1}{6}n(n+1)(4n+5)+2^{n+2}-4$; $\dfrac{(4n)!}{(2n)!\,(n!)^2}$.

A. 2. 1. 32·0 ; $x+1+\frac{1}{x}$. 2. 2 ; $(x-2)(x+3)$.
3. $4n+1$; $3n(2n+1)$; $\frac{4}{3}n(n+2)(2n+5)$.
4. 168 ; 72. 5. $\frac{1}{2}n(n-3)$; $-\frac{1}{6}n(n^2-9n+2)$; $n+1-\frac{1}{n}$.

A. 3. 1. 20 ; 1. 2. 1, $-20\frac{1}{2}$.
3. $(x+y)^5-y^5$; 1·2190. 4. 720.
5. $\{3-2x-(n+3)x^n+(n+2)x^{n+1}\}\div(1-x)^2$;
$\frac{1}{3}n(6n^3+16n^2+6n-1)$.

A. 4. 1. 0·969 ; -1. 3. 1260 ; 180.
4. 6th term ; 1. 5. $\frac{1}{3}n(7n^2-1)$; $\frac{1}{2n}$.

A. 5. 1. (i) increases, $2\sqrt{2}:1$. (ii) $9:2:4$.
2. $6q^2=25r$; $q=-1$. 4. 18(10!).
5. $\frac{(4n)!}{(2n)!(n!)^2}$; $2^{n-3}(n^2+3n)$.

A. 6. 1. $1\frac{1}{3}$; 1·23. 2. $3\frac{1}{3}$, $2\frac{2}{3}$, -6.
3. 36 ; 75. 5. $\frac{(p+q)!}{p!\,q!}$.

A. 7. 1. $a^2=100b^3$; $3^c=10000$; $x=10\,.\,2^y$.
2. 5 ; $5\frac{1}{3}$. 3. 118.
4. $\frac{n(n+2)}{(n+1)^2}$; $\frac{1}{6}n(n-1)(n+7)$.
5. 5th and 6th ; $\frac{3^7.\,5^5.\,7}{8}$; 1, $n>1$.

A. 8. 1. 2·93 ; 2·09, -3·82. 2. $11:5:8$.
3. n^4 ; $p^3r^7=q^9$. 4. $(n-3)!(n-4)(n-5)(n-6)$.
5. $(1+x)^{2n}-(1+x+x^2)^n$.

A. 9. 1. $-(a-b)(b-c)(c-a)$. 2. $1\frac{1}{2}$; 1.
3. $\frac{1}{4}\{pq(p-1)(q-1)+qr(q-1)(r-1)+rp(r-1)(p-1)\}$.
4. $-\frac{1}{4}n^2(2n+3)$; $\frac{1}{4}(n+1)^2(2n-1)$.

A. 10. 1. $4\sqrt{3}-5$; $1\frac{1}{2}$.
3. 246400 ; $\frac{1}{2}(k+1)(k+2)$; $\frac{1}{2}(p+1)(2k-p+2)$. 5. $\frac{1}{n+1}$.

Exercise IV. *a.* (p. 56.)

1. $n>498$. 2. $n>47$. 3. $n>4$.
4. $n>999$; $n>998$. 5. $n>18$; $n>1005$.
6. 1099 ; 902. 7. No. 8. $\to 1$. 9. $\to 2$.

10. $\to 3$.	11. $\to 1$.	12. osc., ± 1.	13. osc., ± 2.
14. $\to \infty$.	15. $\to \infty$.	16. osc., ∞, 0.	17. $\to \infty$.
18. $\to 1$.	19. $\to 0$.	20. $\to 0$.	21. osc., $\pm\infty$.
22. osc., $\pm\infty$.	23. $\to \infty$.	24. osc., 1, 0.	25. $\to 2$.
26. $\to \frac{2}{3}$.	27. $\to 1$.	28. osc., $\pm\infty$.	29. $\to 0$.
30. osc., $\frac{4}{3}$, $-\frac{2}{3}$.	31. osc., $\pm\infty$, -1, 0.		32. $\to \frac{1}{4}$.

Exercise IV. *b*. (p. 59.)

1. n^2; div.   2. $(-1)^{n-1}n$; osc. infin.   3. $\dfrac{2n}{2n+1}$; conv., 1.

4. $\frac{1}{4}\{6n-1+(-1)^n\}$; div.   5. $\frac{10}{9}\{1-(0 \cdot 1)^n\}$; conv., $\frac{10}{9}$.

6. 0, 1, 3 for $n=3p$, $3p+1$, $3p+2$; osc. fin.

7. $\frac{1}{3}\{1-(-2)^n\}$; osc. infin.   8. $\frac{10}{11}\{1-(-0 \cdot 1)^n\}$; conv., $\frac{10}{11}$.

9. 0, $\frac{1}{12}(4n-1)$, $-\frac{1}{12}(4n+1)$ for $n=3p$, $3p+1$, $3p+2$; osc. infin.

10. $\log(n+1)$; div.   11. $\dfrac{n}{6n+4}$; conv., $\frac{1}{6}$.

12. $1-\dfrac{1}{(n+1)!}$; conv., 1.

13. $\dfrac{x(1-x^n)}{1-x}+\dfrac{y(1-y^n)}{1-y}$; $x \geqslant 1$, div.; $x<1$, conv., $\dfrac{x}{1-x}+\dfrac{y}{1-y}$.

14. $1-x^n$; conv., 1; conv., 0; div.

15. $1+x-(1+x)^{1-n}$; (i), (vi) conv., $1+x$; (ii) conv., 0; (iii) div.; (iv) osc. infin.; (v) osc. fin.

16. $\frac{1}{2}$.   17. $\frac{1}{3}$.   18. $\frac{1}{4}$.   19. $(1-x)^{-2}$.   20. $\dfrac{1+x}{(1-x)^3}$.

21. $\{2-(n+1)(n+2)x^n+2n(n+2)x^{n+1}-n(n+1)x^{n+2}\} \div 2(1-x)^3$; $(1-x)^{-3}$.

Exercise IV. *c*. (p. 68.)

1. div.	2. conv.	3. conv.	4. conv.	5. div.
6. conv.	7. div.	8. conv.	9. div.	10. conv.
11. div.	12. conv.	13. div.	14. conv.	15. conv.

16. conv.   17. div.   18. $a<1$, conv.; $a \geqslant 1$, div.

19. $a \leqslant 1$, div.; $a>1$, conv.   20. $a \neq 1$, conv.; $a=1$, div.

21. div.   22. div.   23. div.   24. $p<-1\frac{1}{2}$, conv.; $p \geqslant -1\frac{1}{2}$ div.

Exercise IV. *d*. (p. 71.)

1. $x<1$, conv.; $x \geqslant 1$, div.   2. conv.

3. $x \leqslant 1$, div.; $x>1$, conv.   4. $x \leqslant 1$, conv.; $x>1$, div.

5. $x \leqslant 1$, conv.; $x>1$, div.   6. div.

7. $x<1$, conv.; $x \geqslant 1$, div.   8. div.

9. $x<\frac{1}{3}$, conv. ; $x \geqslant \frac{1}{3}$, div.
10. $x<\frac{3}{2}$, conv. ; $x>\frac{3}{2}$, div. ; [$x=\frac{3}{2}$, div.].
11. $x<1$, conv. ; $x>1$, div. ; [$x=1$, div.]. 12. conv. 13. conv.
14. $x<1$, conv. ; $x \geqslant 1$, div. 15. $x \leqslant 1$, div. ; $x>1$, conv.

Exercise IV. *e*. (p. 77.)

1. conv. 2. conv. 3. osc. fin. 4. conv. 5. div.
6. $x \leqslant -1$, div. ; $|x|<1$, conv. ; $x \geqslant 1$, osc. infin.
7. $|x| \leqslant 1$, conv. ; $|x|>1$, osc. infin.
8. $x<-1$, osc. infin. ; $|x| \leqslant 1$, conv. ; $x>1$, div.
9. Same as No. 8.
10. $x<-\frac{1}{2}$, osc. infin. ; $x=-\frac{1}{2}$, osc. fin. ; $|x|<\frac{1}{2}$, conv. ; $x \geqslant \frac{1}{2}$, div.
11. $x \leqslant -1$ div., $-1<x \leqslant 1$, conv. ; $x>1$, osc. infin.
12. conv. 13. $x<-1$, osc. infin. ; $-1 \leqslant x<1$, conv. ; $x \geqslant 1$, div.
14. $x<-1$, osc. infin. ; $x=-1$, osc. fin. ; $|x|<1$, conv. ; $x \geqslant 1$ div.
15. $x<-1$, osc. infin. ; $x=-1$, conv. if $a \neq 0$, osc. fin. if $a=0$;
 $|x|<1$, conv. ; $x \geqslant 1$, div.
16. Same as No. 14.
17. $x<-1$, div. ; $|x|<1$ conv. ; $x>1$, osc. infin. 18. conv.
19. Sum to infinity of first series = twice that of second.
20. Sum to infinity of first series = $\frac{2}{3}$ that of second.

Exercise V. *a*. (p. 85.)

1. $1+2x+3x^2+4x^3$; $r+1$; $|x|<1$.
2. $1-3x+6x^2-10x^3$; $\frac{1}{2}(r+1)(r+2)(-1)^r$; $|x|<1$.
3. $1+\frac{3}{2}x+\frac{27}{8}x^2+\frac{135}{16}x^3$; $\frac{(2r)!}{(r!)^2}(\frac{3}{4})^r$; $|x|<\frac{1}{3}$, (and $x=-\frac{1}{3}$).
4. $1+5x+25x^2+125x^3$; 5^r ; $|x|<\frac{1}{5}$.
5. $1+8x+40x^2+160x^3$; $\frac{1}{6}(r+1)(r+2)(r+3)2^r$; $|x|<\frac{1}{2}$.
6. $1+2x-2x^2+4x^3$; $2(-1)^{r-1}\frac{(2r-2)!}{(r-1)!\,r!}$; $|x|<\frac{1}{4}$, (and $x=\pm\frac{1}{4}$).
7. $1+\frac{2}{3}x^2+\frac{5}{9}x^4+\frac{40}{81}x^6$; $\frac{2 \,.\, 5 \,.\, 8 \ldots (3p-1)}{p!\,3^p}$ if $r=2p$; $|x|<1$.
8. $1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3$; $3(-1)^r\frac{1 \,.\, 3 \,.\, 5 \ldots (2r-5)}{r!}(\frac{3}{2})^r$;
 $|x|<\frac{1}{3}$, (and $x=\pm\frac{1}{3}$).
9. $2+\frac{1}{4}x^2-\frac{1}{64}x^4+\frac{1}{512}x^6$; $(-1)^{p-1}\frac{(2p-2)!}{(p-1)!\,p!}(\frac{1}{4})^{2p-1}$ if $r=2p$;
 $|x|<2$, (and $x=\pm 2$).
10. $1+nx+\frac{1}{2}n(n+1)x^2+\frac{1}{6}n(n+1)(n+2)x^3$; $\frac{(n+r-1)!}{(n-1)!\,r!}$;
 $|x|<1$, (and $x=1$ if $n<0$, $x=-1$ if $n<1$).

11. $x-x^3+x^5-x^7$; r odd, $(-1)^{\frac{1}{2}(r-1)}$; $|x|<1$.

12. $\frac{1}{a^2}\left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}\right)$; $(r+1)(-a)^{-r-2}$; $|x|<|a|$.

13. $\frac{x}{|a|}\left(1+\frac{x^2}{2a^2}+\frac{3x^4}{8a^4}+\frac{5x^6}{16a^6}\right)$; $\frac{(2p)!}{|a|(p!)^2}\left(\frac{1}{2a}\right)^{2p}$ if $r=2p+1$; $|x|<|a|$.

14. $2\left(2-\frac{1}{6}x^3-\frac{1}{288}x^6-\frac{1}{64.81}x^9\right)$; $(-8)\frac{1.4.7\ldots(3p-5)}{24^p(p!)}$ if $r=3p$;
$|x|<2$, (and $x=\pm 2$).

15. $1+x+\frac{1}{2}(1+n)x^2+\frac{1}{6}(1+n)(1+2n)x^3$; $\frac{(1+n)(1+2n)\ldots(1+rn-n)}{r!}$;
$|x|<\left|\frac{1}{n}\right|$, $\left(\text{and } x=\frac{1}{n} \text{ if } n<0,\ x=-\frac{1}{n} \text{ if } n>1 \text{ or } n<0\right)$.

16. $55(4x)^9$. 17. $\frac{9}{(5x)^{10}}$. 18. $\frac{135}{16}x^3$.

19. $\frac{1155}{4^{10}x^{13}}$. 20. $\frac{(n+4)!}{(n-1)!\,5!}(3x^2)^5$. 21. $-\frac{5}{128}$, (x^4).

22. Two. 23. Four. 24. 5th term. 25. 3rd term.
26. 3rd and 4th terms. 27. 27th and 28th terms. 28. 1st term.
29. 9th and 10th terms. 30. $\frac{8}{11}<x<\frac{10}{13}$. 31. $\frac{2}{3}x^3$.
32. 1·41421. 33. $1-\frac{1}{2}x+\frac{3}{8}x^2+\frac{3}{16}x^3$. 35. $-\frac{4}{5}$.

Exercise V. *b*. (p. 88.)

1. $16=1.2+2.3(\frac{1}{2})+3.4(\frac{1}{2})^2+\ldots$.
2. $\left(\frac{n}{n-1}\right)^n=2+\left(1+\frac{1}{n}\right)\frac{1}{2!}+\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\frac{1}{3!}+\ldots$.
3. $\frac{125}{27}=1+\frac{3}{2}(0{\cdot}8)^2+\frac{3.5}{2.4}(0{\cdot}8)^4+\frac{3.5.7}{2.4.6}(0{\cdot}8)^6+\ldots$.
4. $\frac{1}{8}=\frac{1}{2!}(\frac{3}{8})^2+\frac{1.3}{3!}(\frac{3}{8})^3+\frac{1.3.5}{4!}(\frac{3}{8})^4+\ldots$.

5. 5^6. 6. 4. 7. 2. 8. 3^n. 9. $\frac{1}{3}\sqrt{6}$.
10. $\frac{1}{2}\sqrt[3]{5}$. 11. $\frac{2}{3}\sqrt{3}$. 12. $\frac{1}{4}\sqrt{17}-1$. 13. $\sqrt{3}-1$.
14. $3\sqrt[3]{3}-4$. 15. $10\sqrt[3]{(\frac{10}{7})}-11$. 16. 1. 17. $1-\frac{1}{2}\sqrt[3]{5}$.
18. $\frac{2}{3}(1-2x)^{1\frac{1}{2}}-\frac{2}{3}+2x$. 19. $\{\frac{1}{2}(1+y)\}^n$.
20. x^n. 21. $\frac{1}{2}(1-x)^{-\frac{1}{2}}+\frac{1}{2}(1+x)^{-\frac{1}{2}}$.

Exercise V. *c*. (p. 93.)

1. $\frac{2}{x-1}+\frac{3}{x+4}$. 2. $\frac{2}{x}+\frac{3}{1-x}$. 3. $\frac{3}{x+1}-\frac{5}{4x-1}$.
4. $\frac{3}{x+2}-\frac{5}{(x+2)^2}$. 5. $\frac{2}{x-4}-\frac{1}{x+3}$. 6. $\frac{1}{x+1}-\frac{2}{(x+1)^2}$.

7. $1+\dfrac{1}{x-1}-\dfrac{1}{x+1}$.

8. $\dfrac{1}{3x^2}-\dfrac{1}{9x}+\dfrac{1}{9(x+3)}$.

9. $1+\dfrac{2}{x-2}-\dfrac{1}{x-3}$.

10. $\dfrac{9}{x-3}-\dfrac{5}{x-2}-\dfrac{1}{x-1}$.

11. $\dfrac{1}{x+1}-\dfrac{3}{x-1}+\dfrac{8}{x-2}$.

12. $\dfrac{5}{(x-2)^2}+\dfrac{2}{x-2}-\dfrac{2}{x}$.

13. $\dfrac{1}{x-2}-\dfrac{1}{x+1}+\dfrac{3}{(x+1)^2}$.

14. $\dfrac{2}{(x-1)^2}+\dfrac{1}{2(x-1)}-\dfrac{1}{2(x+1)}$

15. $\dfrac{2}{1-x}+\dfrac{3}{2-x}-\dfrac{6}{(2-x)^2}$.

16. $\dfrac{1}{x-3}+\dfrac{6}{(x-3)^2}+\dfrac{8}{(x-3)^3}$.

17. $\dfrac{2}{x}-\dfrac{2x-1}{x^2+1}$.

18. $x^2+2+\dfrac{1}{x-1}-\dfrac{1}{x+1}$.

19. $\dfrac{2}{x-2}-\dfrac{2}{x+1}+\dfrac{1}{(x+1)^2}+\dfrac{1}{(x+1)^3}$.

20. $\dfrac{28}{x-3}-\dfrac{28}{x-2}-\dfrac{26}{(x-2)^2}-\dfrac{15}{(x-2)^3}$.

21. $\dfrac{2}{x-1}-\dfrac{2x+1}{x^2+x+1}$.

22. $\dfrac{1}{x+2}+\dfrac{3}{x-2}-\dfrac{6}{x+1}+\dfrac{2}{x-1}$.

23. $\dfrac{1}{x+1}-\dfrac{x}{x^2-x+1}$.

24. $\dfrac{2}{(x-1)^3}+\dfrac{2}{(x-1)^2}-\dfrac{1}{x-1}+\dfrac{x-1}{x^2+1}$.

25. $\dfrac{c-b}{x-a}+\dfrac{a-c}{x-b}+\dfrac{b-a}{x-c}$.

26. $\dfrac{1}{(x-a)^2}-\dfrac{1}{(a-b)(x-a)}+\dfrac{1}{(a-b)(x-b)}$.

27. $\dfrac{1}{n+1}=1-{}_nC_1(\tfrac{1}{2})+{}_nC_2(\tfrac{1}{3})-\dots$; $\dfrac{1}{(n+1)(n+2)}=\tfrac{1}{2}-{}_nC_1(\tfrac{1}{3})+\dots$;

$$\frac{3^n(n!)}{4.7\dots(3n+1)}=1-{}_nC_1(\tfrac{1}{4})+{}_nC_2(\tfrac{1}{7})-\dots.$$

29. $\dfrac{2x}{x^2-x+1}-\dfrac{x+3}{x^2+x+1}$.

30. $\dfrac{2}{3n}-\dfrac{7}{6(n+3)}+\dfrac{1}{2(n+5)}$; $\tfrac{359}{360}$.

Exercise V. *d.* (p. 94.)

1. $r(\tfrac{1}{2})^{r+1}$; $|x|<2$.
2. $1-r$; $|x|<1$.
3. $(r+1)^2$; $|x|<1$.
4. 3^r-2^r; $|x|<\tfrac{1}{3}$.
5. $(\tfrac{1}{3})^{r+1}-(\tfrac{1}{4})^{r+1}$; $|x|<3$.
6. $(\tfrac{1}{4})^{r+1}-(-2)^{r+1}$; $|x|<\tfrac{1}{2}$.
7. $\tfrac{1}{3}(r+1)(2r^2+4r+3)$; $|x|<1$.
8. $\tfrac{1}{3}\{2+(-2)^r\}$; $|x|<\tfrac{1}{2}$.
9. $1+2^r$; $|x|<\tfrac{1}{2}$.
10. $(1-3r)(-2)^r-(-\tfrac{1}{2})^r$; $|x|<\tfrac{1}{2}$.
11. $2(-\tfrac{1}{3})^{r+1}-(\tfrac{1}{2})^{r+1}-(\tfrac{1}{4})^{r+1}$; $|x|<2$.
12. $a^{r+1}-b^{r+1}$; $|ax|, |bx|, <1$.
13. $\Sigma a^{r+2}(c-b)$; $|ax|, |bx|, |cx|, <1$.
14. $2(1+3^2x^2+5^2x^4+\dots)$; $6\tfrac{2}{27}=1+3^2(\tfrac{1}{2})^2+5^2(\tfrac{1}{2})^4+\dots$.
17. $1, -1, 0$, if $r=3p, 3p+1, 3p+2$.
18. $(-1)^r, 0, (-1)^{r-1}$, if $r=3p, 3p+1, 3p+2$.

19. $|x|<2$; $|x|>3$. **20.** 3^n.

22. $\frac{(2n)!}{4^n(n!)^2}$, where $n=\frac{1}{2}r$ or $\frac{1}{2}(r-1)$ as r is even or odd.

Exercise V. *e*. (p. 98.)

1. 1·00995. **2.** 9·99667. **3.** 5·03937.

4. 1·25992. **17.** 0·004(5). **18.** $1-\frac{3}{x}+\frac{9}{2x^2}-\frac{13}{2x^3}$.

Exercise V. *f*. (p. 102.)

1. Homog. products, 3 dimensions, 4 letters.

2. Homog. products, 12 dimensions, 3 letters, in each of which a, b, c occur at least once.

3. Homog. products, 8 dimensions, 3 letters, in each of which a occurs at least twice, b at least once.

4. 15 ; 18. **5.** 6 ; 6. **6.** 715. **7.** 56 ; $\frac{(n+k-1)!}{(n-1)!\,k!}$.

8. 20. **9.** 84. **10.** 27. **11.** 21 ; 6. **12.** 462.

13. 155. **14.** 15. **15.** 5151. **16.** $\frac{1}{3}(n+2)(n^2+7n+3)2^{n-1}$.

19. $\frac{(n+r-1)!}{(n-1)!\,r!}$; $\frac{(n+m+2)!}{(n-1)!\,(m+3)!}$ $-\frac{1}{6}n^2(n+1)(n+2)$.

20. $\frac{1}{24}r(r+1)(r+2)(r+3)$; $\frac{1}{2}\Sigma s(r-s+1)(r-s+2)$ for $s=1$ to r.

21. $\frac{3.5.7\ldots(2n-1)}{2.4.6\ldots(2n-2)}$. **23.** $(-1)^{m-1}\frac{(n-2)!}{(m-1)!\,(n-m-1)!}$.

25. $\frac{1}{3}\{10.2^r-(-1)^r-9\}$. **27.** $2(-1)^{m-1}\frac{(n-3)!}{(m-1)!\,(n-m-2)!}$.

Exercise VI. *a*. (p. 108.)

1. 0·693. **2.** 1, $\frac{2}{3}$, $\frac{1}{2}$; $\frac{1}{3}$, $\frac{1}{4}$; $\frac{1}{2}$, $\frac{1}{3}$.

4. 0·406 ; 0·406 ; 2·08. **5.** 1·792 ; 1·792 ; 1·099.

6. 2 ; -1 ; $\frac{1}{2}$. **7.** e^3 ; $\frac{1}{e^2}$; $2e$; e.

8. x ; 2^x ; $1+\log x$. **11.** 0. **12.** $\frac{a}{ax+b}-\frac{f}{fx+g}$.

13. $\log x$; $t\log t+1-t$. **14.** $\frac{1}{e}$. **17.** $\log(ut)=\log t+\log u$.

Exercise VI. *b*. (p. 114.)

1. 0·0198. **2.** 0·40547 ; 1·0986(1) ; 0·47712.

4. 0·08004 ; 2·5649. **6.** 0·69315.

7. $-\frac{1}{r}3^r$; $-\frac{1}{3}\leqslant x<\frac{1}{3}$. **8.** $-\frac{1}{p}(-\frac{1}{4})^p$ if $r=2p$; $|x|\leqslant 2$.

9. $\log 2+\Sigma\left\{-\frac{1}{r}(-\frac{1}{2})^r x^r\right\}$; $-2<x\leqslant 2$. **10.** $-\frac{2}{r}(-1)^r$; $-1<x\leqslant 1$.

11. $-\frac{1}{r}\{1+(-2)^r\}$; $-\frac{1}{2}<x\leqslant\frac{1}{2}$. **12.** $-\frac{1}{r}(-1)^r(1+2^r)$; $-\frac{1}{2}<x\leqslant\frac{1}{2}$.

13. $x+\Sigma\left\{\frac{2r-1}{r-r^2}(-x)^r\right\}$, $r>1$; $-1<x\leqslant 1$.

14. $-x+\frac{3x^2}{2}-\Sigma\frac{2}{r(r-1)(r-2)}x^r$, $r>2$; $-1\leqslant x<1$.

15. $-\Sigma\frac{1}{p}\{2^p x^{2p}+2^{p+1}x^{2p+1}\}$; $|x|<\frac{1}{2}\sqrt{2}$.

16. 0 if r is odd; $\left(\frac{4}{r}\right)(-1)^{\frac{1}{2}r-1}$ if $r=6p$; $\frac{2}{r}(-1)^{\frac{1}{2}r}$ if $r=6p\pm 2$; $|x|\leqslant 1$.

17. $\frac{1}{r}$ if r is odd; 0 if $r=4p$; $\frac{2}{r}$ if $r=4p+2$; $-1\leqslant x<1$.

18. $\frac{1}{r}(-1)^{r-1}$ if $r=3p$; $\frac{2}{r}(-1)^r$ if $r=3p\pm 1$; $-1<x\leqslant 1$.

19. $-\log 2+\Sigma\frac{1}{r}(2^{-r}-1)x^r$; $-1\leqslant x<1$.

20. $-\frac{4}{r}$ if $r=5p$; $\frac{1}{r}$ if $r\neq 5p$; $-1\leqslant x<1$. **21.** $-\frac{1}{r}2^{r+1}$; $-\frac{1}{2}\leqslant x<\frac{1}{2}$.

22. $-\frac{1}{r}2^r$, r even; $\frac{1}{r}(2-2^r)$, r odd; $-\frac{1}{2}\leqslant x<\frac{1}{2}$.

23. $\Sigma(-1)^{r-1}\frac{1}{r}\left(\frac{1}{x}\right)^r$; $\Sigma\frac{1}{r}\left(\frac{1}{x}\right)^{2r}$. **24.** $\Sigma\frac{1}{r(2r-1)}x^{2r}$.

25. $-\frac{1}{r}(\alpha^{-r}+\beta^{-r})$; $|x|\leqslant\frac{1}{2a}\{|b|-\sqrt{(b^2-4ac)}\}$

excluding $x=\frac{1}{2a}\{-b\pm\sqrt{(b^2-4ac)}\}$.

Exercise VI. c. (p. 116.)

1. $\log 2$. **2.** $\log(\frac{4}{3})$. **3.** $\frac{1}{2}\log(\frac{8}{3})$.

4. $-\frac{1}{2}\log(1-x^2)$. **5.** $\log\frac{y+x}{y-x}$. **6.** $\log 2$.

7. $2-\log 4$. **8.** $\log 4-1$. **9.** $\log 2-\frac{1}{2}$.

10. $\left(\frac{1}{x}-1\right)\log(1-x)+1$. **11.** $\frac{1}{4}\left(x-\frac{1}{x}\right)\log\frac{1+x}{1-x}+\frac{1}{2}$.

12. $-\left(1+\frac{1}{x^2}\right)\log(1-x)-\frac{1}{2}-\frac{1}{x}$. **13.** $\frac{1}{2}\log 12$. **14.** $1\frac{1}{2}$.

15. $\frac{1}{4}\log 2$. **16.** $4\log\frac{3}{2}-1$. **18.** $x^2>1$. **19.** $\Sigma\frac{1}{r}y^r$.

21. $\log x$. **22.** $-\frac{1}{n}\left\{2^n-n\,.\,2^{n-2}+\frac{n(n-3)}{1\,.\,2}2^{n-4}-\dots\right\}$.

23. $\frac{1}{3n}\{1+2(-1)^{n-1}\}$.

Exercise VI. *d.* (p. 120.)

3. $a=1, b=\frac{1}{2}$; $a=\frac{1}{2}, b=1, c=\frac{1}{6}$. 5. 1. 6. $\frac{1}{2}$.

7. $\frac{1}{2m}$. 8. $\log_{10}e$. 9. $-\frac{1}{2}$. 10. $\frac{1}{2}$. 11. 1. 12. 1

Exercise VI. *e.* (p. 126.)

1. $1+\Sigma\frac{2^r}{r!}$; $2+\Sigma\frac{2}{(2r)!}$; $1+\Sigma\frac{(-2)^r}{r!}$; $2\Sigma\frac{2^{2r}}{(2r)!}$.

2. 1·64872. 3. 1·1051709. 4. $1-\Sigma\frac{r-1}{r!}x^r$.

5. $1-\Sigma\frac{r-1}{r!}(-x)^r$. 6. $1+\Sigma\frac{(x\log 10)^r}{r!}$.

7. 0, n odd; $\frac{2^{n+1}}{n!}$, n even. 8. $\frac{2^n e^3}{n!}$. 9. $(-1)^n\frac{(n+1)^2}{n!}$.

10. $(-2)^{n-2}\{4a^2-4nab+n(n-1)b^2\}\div n!$.

11. k if $x>2$, $-k$ if $x<2$, where $k=(-\frac{1}{2})^{n-1}\frac{n+1}{n!}$. 12. $\frac{e-1}{e+1}$.

13. $\frac{1}{e}$. 14. e^3-1. 15. $\frac{1}{4}\left(1-\frac{1}{e^4}\right)$. 16. $\frac{1}{8}(e^4-e^{-4})$.

17. $2e$. 18. $e-1$. 19. $\frac{1}{2e}$. 20. $\frac{3e}{2}$. 21. e^2-e.

22. $2\sqrt{e}$. 23. $\frac{3}{e}-1$. 24. $5e$. 25. $12e-3$. 26. $\frac{e^2-3}{8e}$.

27. $\frac{1}{x}(e^x-1-x)$. 28. $\frac{1}{2}(e^{\sqrt{x}}-e^{-\sqrt{x}})\sqrt{x}$. 29. $\frac{3e}{2}$.

30. $-\frac{2}{3}$. 31. $\frac{(x-1)^2}{2x}$. 32. $\frac{1}{x^3}\{(x-1)e^x+1-\frac{1}{2}x^2\}$.

33. 1; $\frac{1}{2}n$. 34. $\frac{e}{r!}$. 35. $\frac{1}{3}n(3n-2)2^{n-3}$.

Exercise VI. *f.* (p. 129.)

2. $1-\frac{1}{2}x+\frac{x^2}{12}-\frac{x^4}{720}$. 3. Error of order x^4.

5. $a-b$. 6. 1. 9. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and $\leqslant 3$.

12. $1+x$. 13. $e\left(1+\frac{1}{2x}-\frac{1}{24x^2}\right)$. 19. $1+\frac{1}{2}p-\frac{p^2}{16}$.

TEST PAPERS A. 11-20. (pp. 130-135.)

A. 11. 1. x^{2q-p-r}. 2. 56 ; 20.

3. $\frac{1}{12}n(4n^2+12n+11)$; $\frac{1}{2}\log\dfrac{n^3}{n^2-1}$.

4. $\dfrac{15}{7(x-3)}-\dfrac{8}{7(x+4)}-\dfrac{1}{x-1}$;

$$\frac{1}{(x-1)^2}+\frac{3}{(x-1)^3}+\frac{3}{(x-1)^4}+\frac{1}{(x-1)^5}.$$

5. 5th term ; $\dfrac{e(1-n)}{n!}$.

A. 12. 1. $1\frac{1}{2}$; $\dfrac{(4n)!\,n!}{2^n(2n)!(2n)!}$. 2. $2(n-2).(n-2)!$.

3. $\dfrac{2n}{3}(2n^2+9n+13)$; $\dfrac{2^n-1}{n!}$.

4. $1+x+\dfrac{x^2}{6}-\dfrac{x^3}{54}+\dfrac{x^4}{216}$; $\frac{1}{2}(3^n+1)$.

5. $4x^2+4x^3$; $4x^3$.

A. 13. 1. $x^2-14x+25=0$.

2. $\dfrac{3}{2(x+1)^2}+\dfrac{1}{4(x+1)}-\dfrac{1}{4(x-1)}$; $\frac{1}{4}\{1+(-1)^n+6(n+1)(-1)^n\}$.

3. div. ; conv. 4. $2^{n+1}-n-2$; $\frac{1}{2}(e-1)^2$

5. $a=0{\cdot}6$, $b=0{\cdot}4$; (i) $\dfrac{2}{n}$, (ii) $\dfrac{2}{n}\{(-1)^{\frac{1}{2}n}-1\}$.

A. 14. 1. $\frac{1}{2}(7+3\sqrt{5})$; 3. 2. 168 ; 72.

3. 6th and 7th terms, $21(\frac{3}{4})^5$; $\sqrt{3}-\frac{2}{3}$. 5. 1.

A. 15. 1. y ; $\sqrt{(n+1)}-1$. 2. $\dfrac{(3n)!}{6(n!)^3}$; $\dfrac{(3n-3)!}{\{(n-1)!\}^3}$.

4. (i) $x<-1$, osc. infin. ; $-1\leqslant x<1$, conv. ; $x\geqslant 1$, div. ; (ii) osc. fin. 5. $\frac{3}{128}$.

A. 16. 1. $-\frac{1}{720}$. 2. $a=3$, $b=c=1$.

3. $\frac{1}{4}n(n+1)(n+4)(n+5)$; $1-\log 2$.

4. $\dfrac{(2n+1)!}{(n!)^2}$; $1-3n$. 5. $0{\cdot}649$.

A. 17. 1. $x^2-bx+ac=0$. 3. 270.

4. $\frac{3}{4}-\log 2$; $2e$. 5. $-\frac{1}{3}$.

A. 18. 1. $1+2(3!)+4(5!)+2(6!)$. 2. 58 ; 120.

3. $3(\frac{1}{2})^n-\frac{2}{3}(\frac{1}{3})^n$. 4. $1+5e$; $2\log 2-1\frac{1}{4}$.

A. 19. 1. (i) $ac>0>ab$; (ii) $a(4a+2b+c)>0>a(4a+b)$.
2. 651.
3. 6, -3, -2, -4, -5, 7.
4. (i) $x<-1$, osc. infin.; $-1\leqslant x<1$, conv.; $x\geqslant 1$, div.; (ii) conv.
5. $n=6p\pm 2$, 0; $n=6p+3$, $-\dfrac{1}{n}$; $n=6p\pm 1$, $\dfrac{2}{n}$; $n=6p$, $-\dfrac{3}{n}$.

A. 20. 1. ac^2, bd^2.
2. 2, 4, 7.
4. Coeff. of x^r is 0 if $r=2p$, is $\dfrac{2}{r}$ if $r=6p\pm 1$, is $-\dfrac{4}{r}$ if $r=6p+3$.

Exercise VII. a. (p. 137.)

1. $x>5$.
2. $x>-\frac{2}{3}$.
3. $x<-1$.
4. min., -19.
5. max., $3\frac{1}{4}$.
6. min., 0.
7. $-\frac{1}{8}$; $c>-\frac{1}{8}$; $c=-\frac{1}{8}$.
8. $2\frac{1}{4}$; $c>2\frac{1}{4}$; $c=2\frac{1}{4}$.
12. $-\dfrac{q}{2p}$, $-\dfrac{2r}{q}$.
15. $a=c$, $b=0$.
17. $a^2-ab+b^2>\frac{3}{4}$.
18. $2\frac{1}{2}-2\sqrt{10}<\lambda<2\frac{1}{2}+2\sqrt{10}$.
19. -2, $\frac{14}{9}$; $a=\frac{7}{8}$, $b=\frac{1}{8}$, $c=\frac{9}{8}$, $d=-\frac{1}{8}$, $p=1$, $q=5$.

Exercise VII. b. (p. 144.)

3. $x<-2$, $x>2$; -2, max.; 2, min.
4. $-1<x<1$; -1, min.; 1, max.
5. All values; none.
6. $x<1$, $x>3$; 1, max.; 3, min.
7. $-2<x<0$, $x>2$; ± 2, min.; 0, max.
8. $-2<x<0$, $x>5$; -2, 5, min.; 0, max.
9. $x=-1$, max.; $x=1$, min.; (i) $f(x)=0$ between -2, -1 and 0, 1 and 1, 2; (ii) $f(x)=0$ at $x=-2$, 1 (repeated); (iii) $f(x)=0$ between -3, -2 only.
10. $f(x)$ steadily increases with x; $f(x)=0$ between 1, 2.
11. $x=2$, max.; $x=6\frac{2}{3}$, min.
12. $x=2$, max.; $x=4\frac{1}{3}$, min.
13. Move Fig. 7 up (i) 2 units, (ii) 27 units, (iii) 76 units. (i) 3 roots; (ii) $x=-1$, 5, 5; (iii) $x=-2$ only.
17. Between -2, -1; -1, 0; 2, 3.
18. Between 2, 3; 4, 5; 5, 6.
19. Between 1, 2; 3, 4; 5, 6.
20. 3.
21. $\frac{1}{2}$, $\frac{1}{2}$, $2\pm\sqrt{5}$.
22. Two equal roots, $\frac{1}{2}$.
24. $4p^6=q^2$.
25. $r\pm 2q\sqrt{q}$; $4q^3>r^2$.

Exercise VII. c. (p. 151.)

1. Min. at $(-2, -\frac{1}{9})$; max. at $(2, -1)$; no value between -1 and $-\frac{1}{9}$.
2. Min. at $(-1, -\frac{1}{2})$; max. at $(1, \frac{1}{2})$; $-\frac{1}{2}\leqslant f(x)\leqslant\frac{1}{2}$.
3. No turning values; like Fig. 10.

4. Min. at $(-3, -\frac{1}{8})$; $f(x) \geqslant -\frac{1}{8}$.

5. Max. at $(3, -\frac{1}{4})$; no value between $-\frac{1}{4}$ and 0.

6. Min. at $(0, 0)$; $0 \leqslant f(x) < 1$; $f(\pm\infty) = 1-$; touches Ox at O.

7. Max. at $(-4, \frac{25}{9})$; min. at $(0, 9)$; $f(-\infty) = 2+, f(\infty) = 2-$; like Fig. 9.

8. No turning values ; $f(-\infty) = 1+, f(\infty) = 1-$.

9. Max. at $\{\frac{1}{2}(-7-\sqrt{39}), \frac{1}{3}(2\sqrt{39}-9)\}$; min. at $\{\frac{1}{2}(-7+\sqrt{39}), \frac{1}{3}(-2\sqrt{39}-9)\}$; $f(-\infty) = 1+, f(\infty) = 1-$; like Fig. 11.

10. $x < -4$, $-1 < x < 1$, $x > 2$; like Fig. 10.

11. $-4-2\sqrt{6}$, $-4+2\sqrt{6}$.

13. $x < -7$, $-2 < x < 3$, $x > 9$; $x < -7$, $\frac{3}{11} < x < 3$.

16. $f(-\infty) = -1+$, $f(\infty) = -1-$. 20. $b^2 > (a+c)^2$.

22. $a^2+c^2 < ab$ requires $b^2 > 4c^2$.

Exercise VIII. *a*. (p. 156.)

1. $x^3-6x^2+11x=6$; $x^4-6x^3+11x^2=6x$;
$x^5-3x^4-5x^3+15x^2+4x=12$; $32x^3-14x=3$.

2. $(x-1)^4(x+1)^3=0$.

3. For equation $x^n+p_1x^{n-1}+\ldots=0$, (i) $p_1=0$; (ii) $p_3=-1$; (iii) $p_1=p_3=0$; (iv) $p_1=p_3=p_5=0$.

5. 12 ; 2 ; 43. 6. 1 ; 7 ; 1. 7. 9. 8. $2\frac{2}{3}$.

9. $-\frac{c}{d}$; $\frac{c^2}{d^2}$. 10. $\frac{b^2}{a^2}$; $\frac{b}{d}$. 11. $2p^2$; $3q$; p^2.

12. p^2-2q ; $-\frac{r}{s}$; $3r-pq$. 13. $2p^2-6q$; $9r-pq$.

14. $27q$; $-2p^3$. 15. r^2-2qs ; $pr-4s$. 16. 3.

Exercise VIII. *b*. (p. 158.)

1. $-1, 2, 5$. 2. $\frac{1}{2}, 2, 8$. 3. $-2, 3, 6$. 4. $-2, 3, 4$.

5. $-\frac{2}{3}, -2, 5$. 6. $\pm 2, 3, -4$. 7. $-2, 3, 3$. 8. $-2, 3, 4$ 6.

9. $-2, 3, 4$. 10. $x^3-2x^2-5x+6=0$; $-2, 1, 3$.

11. $-1, 2, -2$. 12. $2, 3, -4$. 13. $pq=r$. 17. $\frac{a}{\beta}, \frac{\beta}{a}$.

Exercise VIII. *c*. (p. 162.)

1. $2x^3=2x+1$; $x^3+20x^2=2000$; $x^3+5x^2+7x+1=0$.

2. $3x^3=3x^2+1$; $8x^3+6x=3$; $x^3-3x^2+6x=7$.

3. $x^3+3x+3=0$; $x^3=3x^2+9$.

4. $5x^4+3x^3+x^2=1$; $x^4-4x^2+24x=80$.

5. $x^4-13x^2+36=0$; $-4, -3, 1, 2$. 6. $3, 2, -4$.

7. $k=6$, $x^3+19x^2+6x=216$; $\frac{1}{2}, -\frac{2}{3}, -3$.

8. $-1, -1, -2, -2$. 9. $x^3+x+3=0$. 10. $0{\cdot}70, -5{\cdot}70$.
12. $x^4-3px^3+3p^2x^2-p^3x+q=0$. 13. $x^3-3x+2=0$
14. $(m+2)x^2+2(m+6)x=2m-17$; $-2<m<8\frac{1}{2}$.
15. $x^2-x(q^2+2q-2r+2)+(q+r+1)^2=0$.
16. $rx^3+x^2(q+3r)+x(2q+3r)+q+r+1=0$.
17. $x^3+2qx^2+q^2x=r^2$; $rx^3+q^2x^2-2qrx+r^2=0$.
18. $-(a+b+c)$, $-\frac{1}{2}\{a+b+c\pm\sqrt{(a^2+b^2+c^2-2bc-2ca-2ab)}\}$.
19. y^2-2; $-1, -1, 2\pm\sqrt{3}$. 20. $-1, -1, \frac{1}{2}(7\pm3\sqrt{5})$.
21. $3\pm\sqrt{2}, 5$. 23. $2\pm\sqrt{3}, 3\pm2\sqrt{2}$.
24. $2x^3-9x^2+7x+6=0$; $4, 3, \frac{1}{2}$.
25. $-2, 3, 1$; $0, -5, -3$. 27. $a, \frac{a-1}{a}$.
28. $x^3+x^2(3q-p^2)+qx(3q-p^2)+q^3=p^3r$;
(i) $\alpha^2-\beta\gamma=\beta^2-\gamma\alpha=\gamma^2-\alpha\beta$; (ii) $\alpha^2-\beta\gamma=-p\alpha$, etc.

EXERCISE VIII. *d*. (p. 166.)

1. 1·25992. 2. 2·09455.
3. 1·83424, 0·65662, −2·49086.
4. 2·71448, −0·14328, −2·57120. 5. 1·78377.
6. −3·59141. 7. 2·88897, −0·12524, −2·76372.
8. 9·96667. 9. 2·04728, 0·59369.
10. 4·22524. 11. 0·77563, 8·57719, −1·35283.
12. 1·47577. 13. 2·06443(5).
14. 1·35690, 1·69202, −3·04892. 15. 4·58140.

EXERCISE VIII. *e*. (p. 167.)

1. $a+c=2b$. 3. $y^2=4ax$. 4. $x^2=y+2$
5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 6. $x^3+y^3=3axy$.
7. $y^2+ac=cx+by$. 8. $y^3=x^2z$.
9. $a^3-c^3+3d^3=3ab^2$. 10. $a^4b^4+b^4c^4+c^4a^4=a^2b^2c^2d^2$.
11. $(a+c-d)^2=b^2$. 19. $x^2+y^2=a^2+b^2$.
20. $m+n+p=0$. 21. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=2$.

EXERCISE IX. *a*. (p. 171.)

1. −4. 2. −10. 3. −200. 4. −900.
5. −2. 6. −63. 7. 15. 8. 661.
9. −2. 10. $ab-h^2$. 11. $ab(b^2-a^2)$. 12. $-2xy$.

13 $(b-a)^3(b+a)$. 14. $z(x-y)(ps-qr)$.

15. $\frac{1}{13}, -\frac{31}{13}$. 16. $-\frac{11}{4}, -\frac{1}{2}$. 17. $49, -35$.

18. $\frac{31}{11}, \frac{19}{11}$. 19. $\frac{17}{3}, \frac{1}{3}$. 20. $1, 1$.

Exercise IX. *b*. (p. 177.)

1. 2. 2. 4. 3. 0. 4. -1. 5. 0. 6. -18.
7. 0. 8. 0. 9. -15. 10. 0. 11. 0. 12. 0.
13. 0. 14. 0. 15. -3. 16. -8. 17. -6. 18. 0.
19. $2abc$. 20. $(a^3-1)^2$. 21. $3abc-a^3-b^3-c^3$.
22. $abc+2fgh-af^2-bg^2-ch^2$. 23. $a^2+b^2+c^2+1$. 24. 0.
25. $ab+bc+ca=2abc+1$. 26. $a^2+b^2+c^2+1=0$.
27. $ab+bc+ca+1=0$. 28. x^2y^2.
29. $1, 8\pm\sqrt{37}$. 31. 0.

Exercise IX. *c*. (p. 184.)

1. $(a-b)(b-c)(c-a)$. 2. $(a-b)(b-c)(c-a)$.
3. $4abc$. 4. $(x-2y+z)(y-2z+x)(z-2x+y)$.
5. $(a+b+c)(bc+ca+ab-a^2-b^2-c^2)$.
6. $(a-b)(b-c)(c-a)(bc+ca+ab)$.
7. $-(b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2)$.
8. $4abc$. 9. $4, -3, 1$. 10. $3, -2, 2$. 11. $1, 1, 2$.
12. $2, 1, 2$. 13. $1, 1, 1$. 14. $2, 2, 1$.
15. Rows are $a^2+b^2, ac+bd$; $ac+bd, c^2+d^2$. 16. $-b_1\Delta$.
17. 0; rows are $a_1^2+b_1^2, a_1a_2+b_1b_2, a_1a_3+b_1b_3$; $a_1a_2+b_1b_2, a_2^2+b_2^2, a_2a_3+b_2b_3$; $a_1a_3+b_1b_3, a_2a_3+b_2b_3, a_3^2+b_3^2$.
18. $-c(a^2+b^2)$; rows are a^2+bc, ab, ab; $ac, bc+ac, ab$; $c^2, bc, ac+b^2$.
23. $(a-b)(b-c)(c-a)(d-a)(d-b)(d-c)(a+b+c+d)$.
24. $(a-b-c+d)(a-b+c-d)(a+b-c-d)(a+b+c+d)$.

Test Papers A. 21-35. (pp. 186-193.)

A. 21. 1. $h(m-p)=k(m-n)$; $\dfrac{(2n)!}{4^n(n!)^2}$.

2. $3x^2-14x+5=0$. 3. $3n^2+1$; $0{\cdot}670$.

5. $\Sigma\alpha=0$; $\Sigma\alpha+\Sigma\alpha\beta=0$; roots are $\alpha, \dfrac{1}{\alpha}, -1$; roots are $\pm\alpha, \pm\beta$.

A. 22. 1. 0 or 16; -11. 2. $na^3-\frac{1}{6}n(n+1)(2n+1)$; x^n.

3. $c^2(a^2+b^2)=a^4+b^4$. 5. $1-\frac{1}{6}n^2x^2$.

A. 23. 1. $2<x<4$; $4b=3q$, $16c=9r$.

2. $\dfrac{2}{1+x}+\dfrac{1-x}{1+x^2}$; $2+(-1)^r$, $-\{2+(-1)^r\}$.

3. $\frac{1}{3}n(n+1)(4n-1)$; 1·007417. 5. $e-1$.

A. 24. 1. $7-4\sqrt{3}$. 2. $(25a+10h+b)(4a-4h+b)=0$.

4. 8th term. 5. $\dfrac{1-3n+n^2}{n!}$; $-\dfrac{2}{n}$.

A. 25. 2. $x^2+(4r-q^2)(x+r)=0$. 3. $-\frac{1}{3}$, $-\frac{1}{27}$.

4. $\dfrac{1}{a^2}\{1-(n+1)(1-a)^n+n(1-a)^{n+1}\}$; $1-\sqrt[3]{(\frac{1}{3})}$.

5. $\log 2+\frac{1}{2}x^2-\frac{1}{12}x^4$.

A. 26. 1. $-1<x<3$, $x>4$; $p=\frac{1}{2}(a-b-c)$, $q=\frac{1}{2}(b-c-a)$, $r=\frac{1}{2}(c-a-b)$.

2. $\dfrac{2}{x+1}+\dfrac{1}{x-1}-\dfrac{3x-1}{x^2+x+1}$.

3. $\frac{1}{12}n(n+1)(n+2)(3n+5)$; $\dfrac{n}{3n+1}$, $\frac{1}{3}$.

4. -5, -4 ; -2, -1 ; 5, 6. 5. $\dfrac{1}{2\sqrt{e}}$.

A. 27. 1. $c^2-(a-b)^2=2(a+b-c)$; $(a+b+c)^2=b^2-4ac$.

2. 1 if $n=4p$, $4p+1$; 0 if $n=4p+2$, $4p+3$.

3. $\dfrac{1+2x-(3n+1)x^n+(3n-2)x^{n+1}}{(1-x)^2}$; $e-2\frac{1}{2}$.

4. $-\frac{2}{3}$, $-\frac{2}{3}$, $\frac{4}{3}$. 5. $2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$.

A. 28. 1. (ii) $(x-y)(y-z)(z-x)(x+y+z)$. 2. 2880.

4. (i) $\dfrac{n(n+1)}{2(2n+1)(2n+3)}$, $\frac{1}{8}$; (ii) 1.

5. (i) 1, 4, -7 ; 0·3275.

A. 29. 1. (ii) $(y+z)(z+x)(x+y)$. 2. (i) $n-1$; (ii) $n-1+(-1)^{\frac{1}{2}n}$.

3. $\sqrt[3]{2}-1$; $-\dfrac{1}{e}$. 4. $-\frac{11}{24}x^3-\frac{3}{16}x^4$.

A. 30. 1. (ii) $x^3=12x+9$. 2. 480. 3. $\sqrt{3}-\frac{2}{3}$.

4. $-n-1$; $\dfrac{2}{(a-x)^2}+\dfrac{1}{a(a-x)}-\dfrac{a-x}{a(a^2+x^2)}$.

5. $(a^3-b^3)(b^3-c^3)(c^3-a^3)(a^3+b^3+c^3-abcd)=0$.

A. 31. 2. $|x|<1$; $\dfrac{18-13x-x^2}{(1-x)(2+x)(4-x)}$. 3. $\frac{1}{2}$, $\frac{1}{2}$, -2.

4. (i) $\dfrac{27e}{4}$; (ii) $\frac{2}{3}\log 2-\frac{5}{12}$. 5. $-(a+b+c)$, 0.

A. 32. 1. $-2, -7$. 3. (i) $-2, -1$; $1, 2$; $3, 4$; (ii) $2{\cdot}094(5)$.

4. (i) $\dfrac{n(3n+7)}{2(n+1)(n+2)}, \frac{3}{2}$; (ii) $\frac{1}{2}\left(7e+6-\dfrac{1}{e}\right)$. 5. $(a+b+c)^3$.

A. 33. 1. (i) -4 or $\frac{1}{3}$; (ii) $a, \dfrac{3a-4}{2a-3}$. 2. $\dfrac{5}{x-1}+\dfrac{3}{x-3}+\dfrac{3}{x-5}+\dfrac{5}{x-7}$.

5. $\log 2-\frac{1}{4}$.

A. 34. 2. (ii) $\frac{1}{6}(n+1)(n+2)(n+3)$.

3. (i) $x^3-3rx^2+x(3r^2+q^3)=r^3$; (ii) $-2, -1$; $-1, 0$.

4. $2, \frac{2}{27}$. 5. $(a-b)(b-c)(c-a)(d-a)(d-b)(d-c)$.

A. 35. 2. (i) $24abc$; (ii) $(ay-bx)^2+(bz-cy)^2+(cx-az)^2$; $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

3. (i) $\frac{1}{3}n(n+1)(6n^2+14n-5)$; (ii) $3-4\log 2$.